Itay Kaplan, Pierre Simon

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AUTOMORPHISM GROUPS OF FINITE TOPOLOGICAL RANK

ITAY KAPLAN AND PIERRE SIMON

Abstract. We offer a criterion for showing that the automorphism group of an ultrahomogeneous structure is topologically 2-generated and even has a cyclically dense conjugacy class. We then show how finite topological rank of the automorphism group of an ω-categorical structure can go down to reducts. Together, those results prove that a large number of ω-categorical structures that appear in the literature have an automorphism group of finite topological rank. In fact, we are not aware of any ω-categorical structure to which they do not apply (assuming the automorphism group has no compact quotients). We end with a few questions and conjectures.

1. Introduction

Many automorphism groups of Fraïssé structures are known to admit a 2-generated dense subgroup. This is the case for example for dense linear orders and the random graph [Mac86, DM10]. It is however not true that all automorphism groups of say ω-categorical structures have even a finitely generated dense subgroup. For instance a construction of Cherlin and Hrushovski yields an ω-categorical structure whose automorphism groups admits $(\mathbb{Z}/2\mathbb{Z})^\omega$ as a quotient, which implies that it cannot have a finitely generated dense subgroup (see Remark 3.2). However, it seems that the existence of such a large compact quotient is the only known obstruction. We speculate that this might indeed be the case and ask: Let G is the automorphism group of an ω-categorical structure; assume G has no compact quotient, then does it have a finitely generated dense subgroup? This paper is our attempt at answering this question. We fall short of providing a definitive answer, but we succeed in finding sufficient conditions for such a G to admit a finitely generated dense subgroup which seem to apply to all known examples. It is even plausible that those conditions are actually satisfied by all ω-categorical structures with trivial $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, see Conjecture 7.1.

We now describe our main results. We first define a notion of a *canonical independence relation*, or CIR. It is a ternary independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$ which satisfies in particular stationarity over $\emptyset$ and transitivity on both sides. Importantly, we do not assume symmetry. We show that if an ultrahomogeneous structure M admits such a CIR, then it has a 2-generated dense subgroup, and even a cyclically dense conjugacy class (that is, for some $f, g \in G$, the set $\{f^{-n} g f^n \mid n \in \mathbb{Z}\}$ is

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dense). Many, but not all, classical ω-categorical structures have a CIR. Examples that do not include the dense circular order (Corollary 5.7) and the dense infinitely-branching tree (Corollary 6.10). However, an expansion of any of those structures obtained by naming a point does have a CIR (Example 4.18, Remark 5.8). This leads to our second main theorem which completely solves a relative version of our initial question: we show that if $Aut(M)$ has no compact quotients and N is an ω-categorical expansion of M, then if $\mathrm{Aut}(N)$ has a finitely generated dense subgroup, then so does $\mathrm{Aut}(M)$. More precisely, we show that adding two elements from $\mathrm{Aut}(M)$ to $\mathrm{Aut}(N)$ yields a dense subgroup of $\mathrm{Aut}(M)$: see Theorem 5.11.

In Section 6, we give a dynamical consequence of having a CIR. Our main motivation is to relate it to the Ramsey property. We observe in Proposition 3.17 that Ramsey structures admit a weaker form of independence relation. We know by [KPT05] that the Ramsey property is equivalent to extreme amenability of the automorphism group. It is then natural to look for a dynamical interpretation of having a CIR, one goal being to understand to what extent the Ramsey property is not sufficient to imply it. We give a necessary condition for having a CIR which sheds some light on this notion and can be used to prove negative results.

The theorems presented in this paper lead to a number of open questions. In particular, it would be interesting to understand the obstructions to having a CIR and to prove more results about the automorphism groups of structures with a CIR. It is also our hope that some ideas introduced here could be used to develop a general theory of ω-categorical structures. One strategy we have in mind is to show that ω-categorical structures admit *nice* expansions and then to prove *relative* statements which pull down properties from an expansion to the structure itself. See Section 7 for some precise conjectures.

We end this introduction by mentioning some previous work done on this question: The existence of a cyclically dense conjugacy class was shown for the random graph by Macpherson [Mac86], for the Urysohn space by Solecki [Sol05], for dense linear orders by Darji and Mitchell [DM10] and recently for generic posets by Glab, Gordinowicz and Strobin [GGS17]. Kechris and Rosendal [KR07] study the property of having a dense conjugacy class for Polish groups. They show that a number of Polish groups admit cyclically dense conjugacy classes (see Theorem 2.10). Those include the group of homeomorphisms of the Cantor space and the automorphism group of a standard Borel space. Those groups do not fit in our context, although it should be possible to generalize our results so as to include them. In fact, the proof of Theorem 2.10 is very much in the same spirit as the proofs in this paper. Kwiatkowska and Malicki [KM17] give sufficient conditions for an automorphism group G to have a cyclically dense conjugacy class, which gives new examples such as structures with the free amalgamation property and tournaments. Their conditions do not seem to formally imply ours, but all the examples that they give (and in particular structures with free amalgamation) are covered by our theorems. They also show that under the same hypothesis $L_0(G)$ has a cyclically dense conjugacy class. We did not study this.

2. Preliminaries

2.1. ω-categoricity, Ultrahomogeneous structures, Fraïssé limits and model companions. Here we recall the basic facts we need for this paper.

Let L be some first order language (vocabulary).

An L-theory is called *ω-categorical* if it has a unique infinite countable model up to isomorphism. A countable model M is ω-categorical if its complete theory $Th(M)$ is. By a theorem of Engeler, Ryll-Nardzewski and Svenonius, see e.g., [Hod93, Theorem 7.3.1], this is equivalent to saying that $G = \mathrm{Aut}(M)$ is *oligomorphic*: for every $n < \omega$, there are only finitely many orbits of the action of G on M^n. It is also equivalent to the property that every set $X \subseteq M^n$ which is invariant under G is $\emptyset$-definable in M. Also, if M is ω-categorical and A is a finite subset of M then M_A is also ω-categorical, where M_A is the expansion of M for the language L_A which adds a constant for every element in A.

A countable L-structure M is called *ultrahomogeneous* if whenever $f : A \to B$ is an isomorphism between two finitely generated substructures A, B of M, there is $\sigma \in \mathrm{Aut}(M)$ extending f. The *age* of an L-structure M, $\mathrm{Age}(M)$, is the class of all finitely generated substructures which can be embedded into M.

Recall that a class of finitely generated L-structures K closed under isomorphisms has the *hereditary property* (*HP*) if whenever $A \in K$ and $B \subseteq A$ (B is a substructure of A), $B \in K$. The class K has the *joint embedding property* (*JEP*) if whenever $A, B \in K$ there is some C such that both A, B embed into C. It has the *amalgamation property* (*AP*) if whenever $A, B, C \in K$ and $f_B : A \to B$, $f_C : A \to C$ are embeddings, then there is some $D \in K$ and embeddings $g_B : B \to D$, $g_C : C \to D$ such that $g_B \circ f_B = g_C \circ f_C$.

We say that K is *uniformly locally finite* if for some function $f : \omega \to \omega$, for every $A \in K$ and X a subset of A of size n, the structure generated by X has size $\leq f(|X|)$. In the following, "essentially countable" means that K contains at most countably many isomorphism types of structures.

We also recall the notions of model companions and model completions. A theory T is called model complete if whenever $M \subseteq N$ are models of T, $M \prec N$. Suppose that $T_\forall$ is a universal theory. A theory T' is the *model companion* of $T_\forall$ if T' is model complete and $T'_\forall = T_\forall$ (they have the same universal consequences). In other words, every model of $T_\forall$ can be embedded in a model of T'. The theory T' is a *model completion* of $T_\forall$ if in addition it has elimination of quantifiers. Models companions are unique, if they exist. For more, see [TZ12, Section 3.2] and [Hod93, Section 8.3].

Fact 2.1. *Suppose that that K is an essentially countable class of finite L-structures which has HP, JEP and AP.*

(1) [Hod93, Theorem 7.1.2] *The class K has a Fraïssé limit: a unique countable ultrahomogeneous model M with the same age.*
(2) [Hod93, Theorem 7.4.1] *When K is uniformly locally finite, M is ω-categorical and has quantifier elimination.*
(3) *When T is a countable universal L-theory, K is the class of finitely generated models of T and K is uniformly locally finite, then the theory $Th(M)$ is the model completion of T. (See remark below.)*
(4) [Hod93, Theorem 7.1.7] *The converse to (1) also holds: if M is an ultrahomogeneous then the age of M satisfies HP, JEP and AP.*

Remark 2.2. We could not find an explicit reference for (3) (which is well-known), so here is a short argument. Since $Th(M)$ eliminates quantifiers by (2), it is enough to show that $Th(M)_\forall = T_\forall = T$ (since T is universal).

If ψ is universal and $T \models \psi$, then since $\mathrm{Age}(M) \subseteq K$, $M \models \psi$. On the other hand, if $M \models \psi$ where ψ is universal, and $T \not\models \psi$, then there is a model $A' \models T$ such that $A' \models \neg\psi$, and since $\neg\psi$ is existential, the same is true for some finitely generated model $A \subseteq A'$, so $A \in K$ and since $A \in \mathrm{Age}(M)$, we get a contradiction.

Classes K as in Fact 2.1 are called Fraïssé classes or amalgamation classes.

Some examples of Fraïssé limits include DLO (dense linear order), i.e., $Th(\mathbb{Q}, <)$ (here we identify the Fraïssé limit and its theory), the random graph, the random poset (partially ordered set), the random tournament, and more. One example that we will be interested in is that of dense trees.

Example 2.3. Let $L_{dt} = \{<, \wedge\}$, and let $T_{dt,\forall}$ be the universal theory of trees with a meet function $\wedge$. Then $T_{dt,\forall}$ has an ω-categorical model completion by Fact 2.1 (note that the tree generated by a finite set B is just $B \cup \{x \wedge y \mid x, y \in B\}$). We denote the model companion by T_{dt} and call the unique countable model the *dense tree*. See also [Sim15, Section 2.3.1].

Recall that a structure M is *homogeneous* if whenever a, b are finite tuples of the same length, and $a \equiv b$ (which means $\mathrm{tp}(a/\emptyset) = \mathrm{tp}(b/\emptyset)$, i.e., the tuples a, b have the same type), then there is an automorphism taking a to b. Note that ultrahomogeneous structures are homogeneous and the same is true for ω-categorical ones. When M is homogeneous, an *elementary map* $f : A \to B$ for $A, B \subseteq M$ is just a restriction of an automorphism of M.

Finally, we use $\mathfrak{C}$ to represent a monster model of the appropriate theory. This is a big saturated (so also homogeneous) model that contains all the models and sets we will need. This is standard in model theory. For more, see [TZ12, Section 6.1].

2.2. **A mix of two Fraïsé limits.** Suppose that K_1, K_2 are two amalgamation classes of finite structures in the languages L_1, L_2 respectively. Assume the following properties:

(1) The symmetric difference $L_1 \triangle L_2$ is relational.
(2) The class K of finite $L_1 \cup L_2$-structures A such that $A \restriction L_1 \in K_1$ and $A \restriction L_2 \in K_2$ is an amalgamation class. Let M be its Fraïsé limit.
(3) For every $A \in K_1$, there is some expansion A' of A to an $L_1 \cup L_2$-structure such that $A' \restriction L_2 \in K_2$, and similarly that for every $B \in K_2$ there is some expansion B' to $L_1 \cup L_2$ whose restriction to L_1 is in K_1.
(4) If $A \in K$ and $A \restriction L_1 \subseteq B \in K_1$ then there is an expansion B' of B to $L_1 \cup L_2$ such that $A \subseteq B'$ and $B' \restriction L_2 \in K_2$, and similarly for L_2.

Under all these conditions we have the following.

Proposition 2.4. *The structure $M \restriction L_1 = M_1$ is the Fraïsé limit of K_1 and $M \restriction L_2 = M_2$ is the Fraïsé limit of K_2.*

Proof. Start with M_1 (for M_2, the proof is the same). It is enough to show that M_1 is ultrahomogeneous and that $\mathrm{Age}(M_1) = K_1$. The second statement follows from (2), (3) above. For the first, by [Hod93, Lemma 7.1.4] it is enough to show that if $A \subseteq B$ are from K_1 and $f : A \to M_1$ then there is some $g : B \to M_1$ extending f. Using f we can expand A to an $L_1 \cup L_2$-structure A' in such a way that f is an embedding to M (this uses the fact that $L_2 \backslash L_1$ is relational). By (4) we can expand B to an $L_1 \cup L_2$-structure B' such that $A' \subseteq B'$ and $B' \in K$. Since M is the Fraïssé limit of K, it follows by [Hod93, Lemma 7.1.4] again that f can be extended to $g : B' \to M$, and in particular, $g \restriction L_1$ is the embedding we seek. □

2.3. **The maximal compact quotient of the automorphism group of a countable ω-categorical structure.** Assume in this section that M is ω-categorical and countable, and let $G = \mathrm{Aut}(M)$, considered as a topological group in the product topology. The contents of this section are folklore but we give the details for the sake of readability. Recall that for a structure M and $A \subseteq M$, $\mathrm{acl}(A)$ is the set of all algebraic elements over A (elements satisfying an algebraic formula over A: one with finitely many solutions). Similarly, $\mathrm{dcl}(A)$ is the set of all elements definable over A. In the context of ω-categorical structures, $\mathrm{acl}(A)$ and $\mathrm{dcl}(A)$ are defined in terms of the size of the orbit of the action of G fixing A being finite or a singleton respectively. In the next proposition, we describe the maximal compact quotient of $\mathrm{Aut}(M)$ in model theoretic terms. This uses the notion of M^{eq}: the expansion of M obtained by adding a new sort for every $\emptyset$-definable quotient of some $\emptyset$-definable set. See [TZ12, Section 8.4] for more.

For $A \subseteq M$, $\mathrm{Aut}(M/A)$ is the group of automorphisms of M fixing A. This is a closed normal subgroup of G, thus the quotient $G/\mathrm{Aut}(M/A)$ is a Hausdorff topological group (with the quotient topology). We identify $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M^{\mathrm{eq}})$ so that we can put $A = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. In this case, it is also compact as the next proposition says.

Proposition 2.5. *The group $G/\mathrm{Aut}(M/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ is a compact Hausdorff (in fact — profinite) group.*

Proof. Let H be the group of all elementary maps from $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ to $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, also denoted by $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$. The group H is naturally profinite as an inverse system of the family

$$\{H_X \mid X \subseteq M^{\mathrm{eq}},\text{ a finite }\emptyset\text{-definable set}\},$$

where H_X is the group of elementary permutations of X.

Let $\mathrm{res} : G \to H$ be the restriction map $\mathrm{res}(\sigma) = \sigma \restriction \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. We will show that res is onto. Using back-and-forth, it is enough to show that given any complete type $p(x)$ for x in the home sort over $A \cup \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ where $A \subseteq M$ is finite, p can be realized in M. We work in the monster model $\mathfrak{C}$. Let $E = \equiv_{A\cup\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$ be the equivalence relation of having the same type over $A\cup\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Then E is A-invariant and has boundedly many classes in $\mathfrak{C}$. By ω-categoricity, as A is finite E is definable over A, and by compactness, E has finitely many classes. But then for every E-class there must be a representative in M. Since a realization of p must have an E-equivalent element in M, p is realized in M. Note that by compactness we get that every such p is isolated by its restriction to $A \cup X$ where X is some finite $\emptyset$-definable set in M^{eq}.

The kernel of res is precisely $G^0 = \mathrm{Aut}(M/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$, so res induces an isomorphism of groups $G/G^0 \to H$. The group G/G^0 is also a topological group when equipped with the quotient topology. This map is easily seen to be continuous. To see that it is open, it is enough to show that the image of an open neighborhood V of $\mathrm{id}\cdot G^0$ in G/G^0 contains an open neighborhood of id in H. The preimage of V in G is some open set $U \subseteq G$ containing id. Suppose $\mathrm{id} \in U_b = \{\sigma \in G \mid \sigma(b) = b\} \subseteq U$ is some basic open set. As we noted above, there is some finite $\emptyset$-definable set $X \subseteq M^{\mathrm{eq}}$ such that $\mathrm{tp}(b/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ is isolated by $\mathrm{tp}(b/X)$. Then if $\tau \in G$ is such that $\tau \restriction X = \mathrm{id}_X$, then there is some $\sigma \in \mathrm{Aut}(M/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ such that $\sigma\tau(b) = b$, so $\sigma\tau \in U_b$, but then $\tau \in U$ (because U is a union of cosets of $\mathrm{Aut}(M/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$). Hence, the image of V contains the open set $\{\tau \in H \mid \tau \restriction X = \mathrm{id}_X\}$.

Together these two groups are isomorphic as topological groups, so are profinite. □

Definition 2.6. We let $G^0 = \mathrm{Aut}(M/\mathrm{acl}^{\mathrm{eq}}(\emptyset))$.

Proposition 2.7. *If $H \trianglelefteq G$ is normal and closed and G/H is compact then $G^0 \leq H$.*

Proof. Let E_n be the equivalence relation on M^n of having the same H-orbit. Then E_n refines $\equiv$ (having the same type over $\emptyset$) and is definable in M. Indeed, suppose that $\sigma \in G$. Then for every

$a \in M^n$, $\sigma O_H(a) = O_H(\sigma(a))$ (because H is normal), where $O_H(a)$ denotes the orbit of a under H. Hence E_n is invariant under G so $\emptyset$-definable.

In addition, G acts transitively on the H-orbits within each $\equiv$-class by $\sigma \cdot O_H(a) = O_H(\sigma(a))$. The stabilizer of $O_H(a)$ is $\{\sigma \in G \mid \sigma(O_H(a)) = O_H(a)\}$ so open (if σ is there, then to make sure that σ' is there, it is enough that $\sigma'(a) = \sigma(a)$). Note that this action factors through H (i.e., the action $\sigma H \cdot O_H(a) = O_H(\sigma(a))$ is well-defined). Hence, the stabilizer is open in G/H and hence has finite index in G/H, so the number of orbits of H under this action in every $\equiv$-class is finite. By ω-categoricity, the number of $\equiv$-classes (of n-tuples) is finite, so the number of orbits of H is finite.

In summary, E_n is definable and has finitely many classes. Hence these classes belong to $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Given $\sigma \in G^0$, σ fixes the orbits of H under its action on M^n. As H is closed, this means that $\sigma \in H$. □

Corollary 2.8. *The group G/G^0 is the maximal compact Hausdorff quotient of G.*

In light of Corollary 2.8, G *has no compact quotients* (by which we mean that there is no non-trivial compact Hausdorff group which is an image of G under a continuous group homomorphism) iff $G^0 = G$. If N is a normal closed subgroup of G, then we would like to say that (G, N) has no compact quotients iff G/N has no compact quotients. Let us generalize this to any closed subgroup.

Definition 2.9. Suppose that $H \leq G$ is closed. We will say that the pair (G, H) has *no compact quotients* if for all $g \in G$, there is some $h \in H$ such that $g \restriction \mathrm{acl}^{\mathrm{eq}}(\emptyset) = h \restriction \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

Proposition 2.10. *Suppose that $H \leq G$ is closed. Then (G, H) has no compact quotients iff for every closed and normal $N \trianglelefteq G$ such that G/N is compact, $NH = G$.*

Proof. First note that (G, H) has no compact quotient iff $\{gH \mid g \in G\} = \left\{gH \mid g \in G^0\right\}$. This happens iff $G = G^0 H$. Hence the direction from right to left follows by taking $N = G^0$. The direction from left to right is immediate by Proposition 2.7. □

Corollary 2.11. *If $H \leq G$ is closed and normal then (G, H) has no compact quotients iff G/H has no compact quotients as a topological group (i.e., there is no nontrivial compact Hausdorff quotient).*

Proof. Left to right: suppose that G/H has a compact quotient. Then there is a normal closed subgroup $N \trianglelefteq G$ such that $H \leq N$ and G/N is compact. By Proposition 2.7, N contains G^0. Thus, $G^0 H \leq N$ and hence $G = N$ (by Proposition 2.10).

Right to left: for a finite $\emptyset$-definable subset $X \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$, let $G_X^0 = \mathrm{Aut}(M/X) \leq G$ (where we identify G with $\mathrm{Aut}(M^{\mathrm{eq}})$). Note that as X is definable, G_X^0 is normal, hence so is the product with H. Since $\left[G : G_X^0\right]$ is finite, HG_X^0 is closed as a finite union of translates of G_X^0 so that G/HG_X^0 is a compact (even finite) Hausdorff quotient of G/H, so it must be trivial and hence that $HG_X^0 = G$. Take $g \in G$, we need to find $h \in H$ such that $g \upharpoonright \mathrm{acl}^{\mathrm{eq}}(\emptyset) = h \upharpoonright \mathrm{acl}^{\mathrm{eq}}(\emptyset)$. By what we just said, we have that (*) for every finite definable $X \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ there is some $h_X \in H$ such that $g \upharpoonright X = h_X \upharpoonright X$. But since every finite $X \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ is contained in a finite $\emptyset$-definable set $X' \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ (X' is just the union of all conjugates of X), (*) is true for all finite subset $X \subseteq \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.

For every finite tuple a from M, $O_H(a)$ (the orbit of a under H) is a-definable (because H is normal, $g \cdot O_H(a) = O_H(g(a))$ for any $g \in G$, so that $O_H(a)$ is a-invariant thus a-definable by ω-categoricity). In M^{eq}, every definable set X has a code $\ulcorner X \urcorner \in M^{\mathrm{eq}}$ (such that the automorphisms fixing X setwise in M are precisely the automorphisms fixing $\ulcorner X \urcorner$). (This notation is a bit misleading since there could be many possible codes for X.) Let $D \subseteq M^{\mathrm{eq}}$ be the collection of all possible codes $\ulcorner O_H(a) \urcorner$ for all finite tuples a from M. Then D is invariant under G since $g(\ulcorner O_H(a) \urcorner) = \ulcorner O_H(g(a)) \urcorner$ for all $g \in G$ (i.e., $g(\ulcorner O_H(a) \urcorner)$ is a code for $O_H(g(a))$). Let $\bar{c}$ be a tuple enumerating $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Since every $h \in H$ fixes D pointwise, (*) gives us that $\bar{c} \equiv_D g(\bar{c})$ (because to check this equation it is enough to consider finite subtuples). Thus, the map f taking $\bar{c}$ to $g(\bar{c})$ fixing D is an elementary map. By a back-and-forth argument almost identical to the one given in the proof of Proposition 2.5, there is some automorphism $h \in G$ extending f (the point is that the relation $\equiv_{\mathrm{acl}^{\mathrm{eq}}(\emptyset) \cup D \cup A}$ is bounded and A-invariant for any finite set A, hence definable and hence has finitely many classes, all of them realized in M). Since H is closed, and h fixes all H-orbits setwise (as it fixes D), $h \in H$. Finally, $h \upharpoonright \mathrm{acl}^{\mathrm{eq}}(\emptyset) = g \upharpoonright \mathrm{acl}^{\mathrm{eq}}(\emptyset)$ as requested. □

Example 2.12. If M' is an expansion of M and $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$ in M then $(G, \mathrm{Aut}(M'))$ has no compact quotients. This is because in that case, $G^0 = G$.

2.4. **Expansions and reducts of ω-categorical structures.** A group H acts *oligomorphically* on a set X if for all $n < \omega$, the number of orbits of X^n under the action of H is finite for every $n < \omega$.

If M is countable and ω-categorical, and $H \leq G$ is closed, then $H = \mathrm{Aut}(M')$ for some expansion of M. In addition, if H acts oligomorphically on M, then M' is ω-categorical by Ryll-Nardzewski. On the other hand, if $G \leq H$ where H is a closed subgroup of the group of permutations of M, then $H = \mathrm{Aut}(M')$ for some (ω-categorical) reduct M' of M. Two such reducts M', M'' are the same up to bi-definability if they have the same definable sets, which is equivalent to $\mathrm{Aut}(M') = \mathrm{Aut}(M'')$.

Proposition 2.13. *Let M be ω-categorical and $G = \mathrm{Aut}(M)$. Then $G^0 \leq G$ acts oligomorphically on M and $G^0 = \mathrm{Aut}(M')$ for an ω-categorical expansion M' of M with no compact quotients.*

Proof. Let M_0' be the expansion of M^{eq} obtained by naming (i.e., adding constants for) every element in $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. Then M_0' is still ω-categorical (as a many sorted structure) since for any given sort (or finite collection of sorts) S, the equivalence relation $\equiv_{\mathrm{acl}^{\mathrm{eq}}(\emptyset)}$ on S-tuples, of having the same type over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$, is bounded, definable and hence finite, as in the arguments above. Let M' be the reduct to the home sort (so it is also ω-categorical). By definition, $G^0 = \mathrm{Aut}(M')$. Moreover, letting $H = \mathrm{Aut}(M')$, we have that $H^0 = H$. This is because $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{acl}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$. □

2.5. A discussion of Ramsey classes and topological dynamics.

2.5.1. *Ramsey Classes.* Let us start with the definition.

Definition 2.14. For two L-structures A, B, we let $\binom{B}{A}$ be the set substructures of B isomorphic to A. Suppose that K is a class of finite L-structures. We say that K is a *Ramsey class* if for every $A, B \in K$ and $k < \omega$ there is some $C \in K$ such that $C \to (B)^A_k$: for every function $f : \binom{C}{A} \to k$ there is some $B' \in \binom{C}{B}$ such that f is constant on $\binom{B'}{A}$.

Say that an L-structure M is a *Ramsey structure* if $\mathrm{Age}(M)$ is a Ramsey class.

Ramsey classes are extremely important classes of finite structure. There are many examples of Ramsey classes, in particular the class of finite linear orders (this is just Ramsey's theorem) and furthermore, by a theorem of Nešetřil and Rödl [NR83, Theorem A], proved independently by Harrington and Abramson [AH78, Appendix B], the class of all finite linearly ordered graphs, or more generally the class of all finite linearly ordered structures in a fixed finite relational language is Ramsey. In fact, [HN16, Theorem 4.26] generalizes this to allow function symbols as well.

The dense tree is an important example for us. However it is not a Ramsey structure due to the following fact.

Fact 2.15. [Bod15, Corollary 2.26] *If M is an ω-categorical Ramsey structure, then there is a definable linear order on M.*

Adding a generic linear order to a dense tree will not result in a Ramsey structure. By this we mean the model completion of the theory $T_{dt,<,\forall}$ in the language $\{<, \wedge, <'\}$ which says that the $\{<, \wedge\}$-part is a meet tree, and $<'$ is a linear order. The class of finite structures of $T_{dt,<,\forall}$ easily has HP, JEP and HP, thus this model completion exists (see Fact 2.1). Call its Fraïssé limit the generically linearly ordered tree. It turns out that this structure is not Ramsey, see Claim 2.18. In any case, we can add a linear order to the tree structure and make it Ramsey.

Example 2.16. ([Sco15, Corollary 3.17], and see there for more references) Let $L = \{<, <_{\mathrm{lex}}, \wedge\}$ and let M be the L-structure whose universe is the tree $\omega^{<\omega}$ with the natural interpretations of $<$ as the tree order, $\wedge$ as the meet function ($s \wedge t = s \upharpoonright \mathrm{len}(s \wedge t)$ where $\mathrm{len}(s \wedge t) = \max\{k \mid s \upharpoonright k = t \upharpoonright k\}$), $<_{\mathrm{lex}}$ as the lexicographical order ($s <_{\mathrm{lex}} t$ iff $s < t$ or $s(\mathrm{len}(s \wedge t)) < t(\mathrm{len}(s \wedge t))$). Then M is a Ramsey structure.

Fact 2.17. [Neš05, Theorem 4.2] *A Ramsey class that has HP and JEP has AP (see Section 2.1 for the definitions).*

It is easy to see that $K = \mathrm{Age}(\omega^{<\omega})$ in the language $\{<, <_{\mathrm{lex}}, \wedge\}$ has JEP, and thus we conclude that it has AP. Let M be its Fraïssé limit and let $T_{dt,<_{lex}} = Th(M)$. Since K is uniformly locally finite it follows that M is ω-categorical and $T_{dt,<_{lex}}$ has quantifier elimination (see Fact 2.1).

By Proposition 2.4 we have that the restriction of M to $\{<, \wedge\}$ is a dense tree, i.e., a model of T_{dt} (see Example 2.3). Here, $L_1 = \{<, \wedge\}$, $L_2 = \{<, <_{\mathrm{lex}}, \wedge\}$, K_1 the class of finite meet trees and $K_2 = \mathrm{Age}(\omega^{<\omega})$. Similarly, the restriction of the generically linearly ordered tree is also a dense tree.

Claim 2.18. The generically linearly ordered tree is not a Ramsey structure.

Proof. Let $K = \mathrm{Age}(M)$ where M is the countable generically linearly ordered tree. As $N = M \upharpoonright \{<, \wedge\} \models T_{dt}$, by ω-categoricity, there is some linear order $<_{lex}$ such that $(N, <_{lex}) \models T_{dt,<_{lex}}$.

Fix some $A \in K$ whose universe contains 3 elements $a, b, a \wedge b$ such that $a \wedge b < a, b$ and $a <' b <' a \wedge b$. Define a coloring $f : \binom{M}{A} \to 2$ by $f(A') = 0$ iff $[a <' b$ iff $a <_{lex} b$ (in A')$]$. If M were Ramsey, there would be some homogeneous $B' \in \binom{M}{B}$ where $B \in K$ is such that B contains 5 elements $a, b, a \wedge b, c, a \wedge c$ where $a \wedge c < a, c$ and $a \wedge c < a \wedge b < a, b$ and $a <' c <' b <' a \wedge b <' b \wedge c$. It follows that for any copy of B in M, $a <_{lex} c$ iff $b <_{lex} c$. However in B' we have that $a <_{lex} c$ iff $c <_{lex} b$ — contradiction. □

We end this discussion with the following fact.

Fact 2.19. [Bod15, Theorem 3.10] *If M is an ultrahomogeneous Ramsey structure, and $c \in M$, then the structure $M' = (M, c)$ where c is a named constant is still Ramsey (and ultrahomogeneous).*

2.5.2. *Topological dynamics and extremely amenable groups.* Let us first recall some basic notions from topological dynamics.

Suppose that G is a topological group. A *G-flow* is a compact Hausdorff space X with a continuous action of G. A subflow of X is a compact subspace $Y \subseteq X$ that is preserved by the action of G, i.e., $gY = Y$ for all $g \in G$. A *G-ambit* is pair (X, x_0) where X is a G-flow and x_0 has a dense orbit. A universal G-ambit is a G-ambit (X, x_0) such that for any ambit (Y, y_0) there is a map $f : X \to Y$ taking x_0 to y_0 that commutes with the action: $gf(x) = f(gx)$ for all $x \in X$ (it follows that f is onto). A universal G-ambit exists and is unique (see [Aus88, Chapter 8]). Finally, G is called *extremely amenable* if for every G-flow X, there is some fixed point $x \in X$ (i.e., $gx = x$ for all $g \in G$).

Kechris, Pestov, and Todorcevic [KPT05] found a striking link between Ramsey classes and topological dynamics, described in the following theorem.

Fact 2.20. [KPT05, Theorem 4.7] *Suppose that M is a countable ultrahomogeneous linearly ordered structure in a countable language. Then* $\mathrm{Aut}(M)$ *is extremely amenable iff M is a Ramsey structure.*

3. Having a cyclically dense conjugacy class

Definition 3.1. Suppose that G is a topological group.

(1) The group G has *finite topological rank* if it has a finitely generated dense subgroup. Similarly, G has *topological rank n* (or *topologically n-generated*) if there are $\{f_i \mid i < n\} \subseteq G$ which generate a dense subgroup.
(2) The group G has a *cyclically dense conjugacy class* if there are $f_1, f_2 \in G$ such that $\left\{f_1^{-n} f_2 f_1^n \,\middle|\, n \in \mathbb{Z}\right\}$ is dense in G.

Remark 3.2. If $f : G_1 \to G_2$ is a surjective continuous homomorphism, and G_1 has a dense conjugacy class, then so does G_2. Also, if H is a finite nontrivial group (with the discrete topology), then H cannot have a dense conjugacy class. Therefore, the same is true for nontrivial profinite groups. Hence if G is a topological group with a nontrivial profinite quotient, it does not contain a dense conjugacy class. Let M be countable and ω-categorical and $G = \mathrm{Aut}(M)$. By Proposition 2.5, G/G^0 is profinite. It follows that one constraint against having a dense conjugacy class is having a nontrivial compact quotient. In model theoretic terms, it means that if $\mathrm{acl}^{\mathrm{eq}}(\emptyset) \neq \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$ (equivalently, $G^0 \neq G$), then G cannot have a dense conjugacy class.

Moreover, in general, we can have that $\left((\mathbb{Z}/2\mathbb{Z})^{\omega}, +\right)$ is a quotient of $\mathrm{Aut}(M)$, which is locally finite so certainly not topologically finitely generated, in which case $G = \mathrm{Aut}(M)$ cannot be topologically finitely generated. For example, let $L = \{E_n \mid n < \omega\}$ where each E_n is a $2n$-ary relation. Let $T_\forall$ say that E_n is an equivalence relation with two classes, and that $(x_1, \ldots, x_n)\ E_n\ (y_1, \ldots, y_n) \to \bigwedge_{i \neq j} x_i \neq x_j \wedge y_i \neq y_j$ (there is no relation between different E_n's). The class of finite $T_\forall$ models has AP and JEP (and it is essentially countable, see above Fact 2.1), so there is a model completion T. Then T is ω-categorical by quantifier elimination (only finitely many definable sets in a given arity). Let M be the countable model. Then $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ contains M^n/E_n for all $n < \omega$, and for any $\eta \in \mathbb{Z}/2\mathbb{Z}$, there is an automorphism $\sigma \in G = \mathrm{Aut}(M)$ such that $\sigma \restriction M^n/E_n$ is the identity iff $\eta(n) = 0$. In fact one can show that $G/G^0 = \left((\mathbb{Z}/2\mathbb{Z})^{\omega}, +\right)$ and that σ fixes $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$ iff it fixes all E_n-classes.

This construction is due to Cherlin and Hrushovski (see [Hod89, Proof of Theorem 5.2] and [Las82, Addendum (2)]). Using a similar technique, in [EH90, Lemma 3.1] it is shown that any profinite group H which has a countable basis of open subgroups can be realized as G/G^0 for some automorphism group G of an ω-categorical structure M.

Definition 3.3. Suppose that M is some structure and $a, b \in M$ are some tuples. We write $a \mathop{\smile\!\!\!\!\!\!|}^{ns} b$ to say that $\operatorname{tp}(a/b)$ does not split over $\emptyset$. When M is homogeneous, this means, letting B be the set b enumerates: if $g : B' \to B''$ is a partial automorphism of B (i.e., $B', B'' \subseteq B$ and g extends to an automorphism of M) then g extends to an automorphism of M which fixes a pointwise.

In the next definition, our convention is that for sets A, B, we write $A \mathop{\smile\!\!\!\!\!\!|} B$ if this is true for tuples enumerating A, B.

Definition 3.4. An automorphism $\sigma \in \operatorname{Aut}(M)$ is *repulsive* if for every finite set $A \subseteq M$ there is some n such that $A \mathop{\smile\!\!\!\!\!\!|}^{ns} \sigma^n(A)$ and $\sigma^n(A) \mathop{\smile\!\!\!\!\!\!|}^{ns} A$. Say that σ is *strongly repulsive* if this is true for all $m \geq n$ as well.

Suppose that M is some structure. For $k < \omega$, add predicates $P_1, \ldots, P_k$ to the language, and let $\bigsqcup_k M$ be the disjoint union of k copies of M, one for each predicate, where each copy has the same structure as M. Then $\operatorname{Aut}(\bigsqcup_k M) = \operatorname{Aut}(M)^k$.

Proposition 3.5. *If σ is a (strongly) repulsive automorphism of an L-structure M, then $\sigma^{\times k} \in \operatorname{Aut}(M)^k$ is a (strongly) repulsive automorphism of the structure $\bigsqcup_k M$ for all $k < \omega$.*

Proof. Suppose that $A \subseteq \bigsqcup_k M$ is finite. Then we may assume, enlarging A, that $A = \bigsqcup_k A_0$ for some finite $A_0 \subseteq M$ (i.e., the disjoint union of the same set in the different predicates). Thus the proposition follows from the fact that if $A_0 \mathop{\smile\!\!\!\!\!\!|}^{ns} B_0$ in M, then $\bigsqcup_k A_0 \mathop{\smile\!\!\!\!\!\!|}^{ns} \bigsqcup_k B_0$ in $\bigsqcup_k M$, which is clear. □

A repulsive automorphism is a special case of a topologically transitive map:

Definition 3.6. Suppose that X is a topological space. A map $f : X \to X$ is called *topologically transitive* if for every two nonempty open sets $U, V \subseteq X$, there is some $n < \omega$ such that $f^n(U) \cap V \neq \emptyset$.

Lemma 3.7. *Suppose that M is a countable structure and that $\sigma \in G = \operatorname{Aut}(M)$ is repulsive. Then conjugation by σ in G is topologically transitive.*

Proof. Denote by $f : G \to G$ the conjugation by σ. Suppose that U, V are two nonempty basic open subsets of G, i.e., $U = U_{a,b} = \{\tau \in G \mid \tau(a) = b\}$ (where a, b are finite tuples) and $V = U_{c,d}$. Note that $f(U_{a,b}) = U_{\sigma(a),\sigma(b)}$. So we need to find some $n < \omega$ such that $U_{\sigma^n(a),\sigma^n(b)} \cap U_{c,d}$ is nonempty. This means that we need to show that for some $n < \omega$, $\sigma^n(a)\, c \equiv \sigma^n(b)\, d$. As σ is repulsive, there is some $n < \omega$ such that $cd \mathop{\smile\!\!\!\!\!\!|}^{ns} \sigma^n(ab)$ and $\sigma^n(ab) \mathop{\smile\!\!\!\!\!\!|}^{ns} cd$. Since both $U, V \neq \emptyset$, $a \equiv b$ and $c \equiv d$, and thus $\sigma^n(a)\, c \equiv \sigma^n(b)\, c \equiv \sigma^n(b)\, d$. □

Fact 3.8. [Sil92, Proposition 1.1] *If X is second countable (i.e., have a countable basis) of second category (i.e., not meager) and separable, and $f : X \to X$ is topologically transitive, then for some $x \in X$, $\{f^n(x) \mid n < \omega\}$ is dense.*

In fact, it follows from the proof there that the set of such x's is comeager.

Proof. Since it is not written explicitly in [Sil92], we provide a proof of the last statement (based on the proof from there). Consider the set F of $x \in X$ such that $\{f^n(x) \mid n < \omega\}$ is not dense. Fix some countable basis $\mathcal{V}$ of open sets. For each $x \in F$, there is some $U_x \in \mathcal{V}$ such that $f^n(x) \notin U_x$ for all n. Now, $\bigcup \{f^{-n}(U_x) \mid n < \omega\}$ is open and dense since f is topologically transitive. Hence, its complement A_{U_x} is closed, nowhere dense and contains x. The union $\bigcup \{A_{U_x} \mid x \in F\}$ is a countable union which contains F, hence F is meager. □

In our case, $G = \mathrm{Aut}(M)$ for a countable model M is of second category, since it is even Polish. Hence we immediately get the following corollary.

Corollary 3.9. *Suppose that M is a countable structure. Suppose that $G = \mathrm{Aut}(M)$ contains a repulsive automorphism σ. Then G is topologically 2-generated and moreover has a cyclically dense conjugacy class, and even: the set of τ for which $\{\sigma^n \tau \sigma^{-n} \mid n \in \mathbb{N}\}$ is dense is comeager.*

An alternative, more direct proof is as follows. Assume that σ is a repulsive automorphism, and construct $\tau \in \mathrm{Aut}(M)$ by back-and-forth, so that $\{\sigma^{-n} \tau \sigma^n \mid n \in \mathbb{Z}\}$ is dense in G. We leave the details as an exercise. We also point out that according to [KM17, Lemmas 5.1 and 5.2], the fact that this set of τ is comeager actually follows immediately from the fact that there is one such τ.

Now we turn to the question of finding a repulsive automorphism.

Definition 3.10. Suppose that M is a countable structure. A ternary relation $\mathop{\smile\!\!\!\!\!\!|}$ on finite subsets of M, invariant under $\mathrm{Aut}(M)$ is a *canonical* independence relation (CIR) if it satisfies the following properties for all finite sets A, B, C, D:

- (Stationarity over $\emptyset$) If $A \mathop{\smile\!\!\!\!\!\!|} B$, $A' \mathop{\smile\!\!\!\!\!\!|} B'$, a, b, a', b' are tuples enumerating A, B, A', B', and $a \equiv a'$, $b \equiv b'$ then $ab \equiv a'b'$. Note that this (together with monotonicity, see below) implies non-splitting: if $A \mathop{\smile\!\!\!\!\!\!|} B$ then $A \mathop{\smile\!\!\!\!\!\!|}^{ns} B$ and $B \mathop{\smile\!\!\!\!\!\!|}^{ns} A$ (where $A \mathop{\smile\!\!\!\!\!\!|} B$ means $A \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} B$).
- (Extension (on the right)) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ then for all finite tuples d there is some $d' \equiv_{BC} d$ such that $A \mathop{\smile\!\!\!\!\!\!|}_C B \cup d'$.
- (Transitivity on both sides) If $A \mathop{\smile\!\!\!\!\!\!|}_{DC} B$ then if $D \mathop{\smile\!\!\!\!\!\!|}_C B$ then $AD \mathop{\smile\!\!\!\!\!\!|}_C B$ and if $A \mathop{\smile\!\!\!\!\!\!|}_C D$ then $A \mathop{\smile\!\!\!\!\!\!|}_C BD$ (DC means $D \cup C$).
- (Monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A' \subseteq A$, $B' \subseteq B$ then $A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.
- (Existence) $A \mathop{\smile\!\!\!\!\!\!|}_C C$ and $C \mathop{\smile\!\!\!\!\!\!|}_C A$.

For finite tuples a, b, c enumerating sets A, B, C respectively, write $a \mathop{\smile\!\!\!\!\mid}_c b$ for $A \mathop{\smile\!\!\!\!\mid}_C B$.

Note that we do not ask for symmetry nor for base monotonicity (if $A \mathop{\smile\!\!\!\!\mid}_C BD$ then $A \mathop{\smile\!\!\!\!\mid}_{CD} B$).

We say that a CIR is *defined on finitely generated substructures* if for all finite $A, B, C, A', B', C' \subseteq M$, if $\langle A\rangle = \langle A'\rangle$, $\langle B\rangle = \langle B'\rangle$ and $\langle C\rangle = \langle C'\rangle$ then $A \mathop{\smile\!\!\!\!\mid}_C B$ iff $A' \mathop{\smile\!\!\!\!\mid}_{C'} B'$. ($\langle A\rangle$ is the substructure generated by A).

Remark 3.11. If a CIR is defined on finitely generated substructures, then it naturally induces a relation $\mathop{\smile\!\!\!\!\mid}^*$ whose domain is finitely generated substructures by setting $\langle A\rangle \mathop{\smile\!\!\!\!\mid}^*_{\langle C\rangle} \langle B\rangle$ iff $A \mathop{\smile\!\!\!\!\mid}_C B$. The relation $\mathop{\smile\!\!\!\!\mid}^*$ satisfies the natural variants of Definition 3.10. For example, transitivity to the left becomes: for all finitely generated substructures A, B, C, D, if $A \mathop{\smile\!\!\!\!\mid}^*_{\langle DC\rangle} B$ and $D \mathop{\smile\!\!\!\!\mid}^*_C B$ then $\langle AD\rangle \mathop{\smile\!\!\!\!\mid}^*_C B$. Similarly, extension becomes: if $A \mathop{\smile\!\!\!\!\mid}^*_C B$ then for all finite tuples d there is some $d' \equiv_{BC} d$ such that $A \mathop{\smile\!\!\!\!\mid}^*_C \langle Bd'\rangle$.

On the other hand, if we have a relation $\mathop{\smile\!\!\!\!\mid}^*$ satisfying these natural properties on finitely generated substructures of M, then there is also a CIR defined on finitely generated substructures: define $A \mathop{\smile\!\!\!\!\mid}_C B$ iff $\langle A\rangle \mathop{\smile\!\!\!\!\mid}^*_{\langle C\rangle} \langle B\rangle$.

Theorem 3.12. *Assume that* $\mathop{\smile\!\!\!\!\mid}$ *is a CIR defined on finitely generated substructures of an ultrahomogeneous structure* M*. Then there is a strongly repulsive automorphism in* $\mathrm{Aut}(M)$*.*

Proof. Let S be the set of all closed nonempty intervals of integers, i.e., sets of the form $[i, j]$ for $i \leq j$ from $\mathbb{Z}$. For every finite set $s \in S$ we attach a countable tuple of variables $\bar{x}_s = \langle x_{s,i} \mid i < \omega\rangle$ in such a way that if $t \neq s \in S$ then $\bar{x}_s \cap \bar{x}_t = \emptyset$. For $s \in S$, let $\bar{y}_s = \bigcup \{\bar{x}_t \mid t \in S, t \subseteq s\}$, and $\bar{y} = \bigcup \{\bar{x}_s \mid s \in S\} = \bigcup \{\bar{y}_s \mid s \in S\}$. For a $\bar{y}$-tuple ($\bar{y}_s$-tuple) $\bar{a}$ and $t \in S$ (contained in s), we write $\bar{a} \upharpoonright t$ for $\bar{a} \upharpoonright \bar{y}_t$ and similarly for a type in $\bar{y}$ ($\bar{y}_s$).

By Fact 2.1, the age of M, denoted by K, has HP, JEP and AP. Fix an enumeration $\langle (A_l, B_l) \mid l < \omega\rangle$ of all pairs $A, B \in K$ such that $A \subseteq B \subseteq M$, including the case $A = \emptyset$.

For every $1 \leq n < \omega$, we construct a complete quantifier free type $r_n\left(\bar{y}_{[0,n-1]}\right) \in S^{\mathrm{qf}}(\emptyset)$ and a sequence $\langle f_{n,l,i} \mid l, i < \omega\rangle$ such that:

(1) If $\bar{a} \models r_n$ then $\bar{a}$ is enumerates a finitely generated substructure $A \in K$ and for a fixed $l < \omega$, $\langle f_{n,l,i} \mid i < \omega\rangle$ enumerates a countable set of functions that contains all embeddings of A_l into A. (Formally, $f_{n,l,i}$ is a function from A_l into the variables $\bar{y}_{[0,n-1]}$ which the type r_n "thinks" is an embedding.)

(2) If $0 < m < n$ then for every interval $s \in S$ with $s \subseteq n$ such that $|s| = m$, $r_n \upharpoonright s = r_m(\bar{y}_s)$.

(3) If $\bar{a} \models r_n$ then for every $0 < m < n$, every pair from $\{(A_l, B_l) \mid l < n-1\}$ and every $f \in \{f_{m,l,i} \mid l, i < n-1\}$ (so f is an embedding of A_l into the structure enumerated by $\bar{a} \upharpoonright [0, m-1]$), there is an embedding g of B_l into the structure enumerated by $\bar{a}$ such that g extends f.

(4) If s,t are two intervals contained in n such that $\min s \leq \min t$ and $\bar{a} \models r_n$ then $\bar{a} \upharpoonright s \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright s \cap t} \bar{a} \upharpoonright t$ (see Remark 3.11).

How?

For $n = 1$, let $r_1(\bar{y}_{\{0\}})$ be a complete quantifier free type of a tuple enumerating some $D_0 \in K$. Note that it trivially satisfies all the assumptions.

Suppose we found r_n satisfying all the properties and we construct r_{n+1}. Let $\bar{a} \models r_n$ from M. Let $f : \bar{a} \upharpoonright [0, n-2] \to \bar{a} \upharpoonright [1, n-1]$ be defined by setting $f(a_{s,i}) = a_{s+1,i}$ for all $s \in S$ contained in $[0, n-2]$ (in general, $s+n$ is the translation of s by n). It is an isomorphism since both $\bar{a} \upharpoonright [0, n-2]$ and $\bar{a} \upharpoonright [1, n-1]$ realize r_{n-1}. By the homogeneity of M, we can extend $\bar{a} \upharpoonright [1, n-1]$ to some tuple $\bar{a}'$ enumerated by $\bar{y}_{[1,n]}$, and extend f to an isomorphism $f' : \bar{a} \to \bar{a}'$ such that $f'(a_{s,i}) = a_{s+1,i}$ for all $s \in S$ contained in $[0, n-1]$. (When $n = 1$, $f = \emptyset$ and $\bar{a}'$ is just an isomorphic copy of $\bar{a}$.)

By existence, we have that $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright [1,n-1]} \bar{a} \upharpoonright [1, n-1]$. By extension, we may assume that $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright [1,n-1]} \bar{a}'$. Let $\bar{a}''$ be a $\bar{y}_{n+1}$-tuple containing $\bar{a}\bar{a}'$ that contains witnesses to all the relevant pairs and embeddings from (3), ordered in such a way that $\bar{a}$ enumerates the $\bar{y}_n$-part and $\bar{a}'$ enumerates the $\bar{y}_{[1,n]}$-part (so the remaining parts of $\bar{a}''$ are enumerated by $\bar{x}_{[0,n]}$). This can be done since K has AP and JEP. Let $r_{n+1} = \mathrm{tp}(\bar{a}'')$.

Now we have to check that (1)–(4) hold. We prove this by induction on n.

(1) and (3) are clear by construction and the induction hypothesis. (2) follows by the choice of f and f'.

Let us prove (4), so fix $\bar{a} \models r_{n+1}$. By monotonicity it is enough to prove that $\bar{a} \upharpoonright [0, k-1] \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright [m,k-1]} \bar{a} \upharpoonright [m, n]$ for any $m \leq n, 1 \leq k \leq n+1$. We may assume that $1 \leq m$ and $k \leq n$ (otherwise this is true by existence).

Note that $\bar{a} \upharpoonright [0, k-1] \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright [1,n-1]} \bar{a} \upharpoonright [m, n]$ by construction (and monotonicity). By induction, $\bar{a} \upharpoonright [1, n-1] \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright [m,n-1]} \bar{a} \upharpoonright [m, n]$. Hence by transitivity and monotonicity we have that $\bar{a} \upharpoonright [0, k-1] \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright [m,n-1]} \bar{a} \upharpoonright [m, n]$. By induction we have that $\bar{a} \upharpoonright [0, k-1] \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright [m,k-1]} \bar{a} \upharpoonright [m, n-1]$. By applying transitivity (and monotonicity) again, we get that $\bar{a} \upharpoonright [0, k-1] \mathop{\smile\!\!\!\!\!\!\mid}^{*}_{\bar{a} \upharpoonright [m,k-1]} \bar{a} \upharpoonright [m, n]$. This finishes the proof of (4).

By compactness, we can find a $\bar{y}$-tuple $\bar{b}$, such that for every $s \in S$, $\bar{b} \upharpoonright s \models r_{|s|}$. Note that $\bar{b}$ enumerates a Fraïssé limit N by (3). This is as in the proof of [Hod93, Theorem 7.1.2]. More precisely, by JEP, the age of N is K (given $A \in K$, by JEP there is some B such that (D_0, B) is one of (A_l, B_l) and $A \subseteq B$, so this is taken care of in the construction). In addition, by [Hod93, Lemma 7.1.4 (b)], N is ultrahomogeneous.

Hence by uniqueness of the Fraïssé limit (see Fact 2.1), we may assume that $\bar{b}$ enumerates M. Let $\sigma : M \to M$ be defined by $\sigma(b_{s,i}) = b_{s+1,i}$. By construction, σ is an automorphism. Now, any finite subset of M is contained in $\bar{b} \upharpoonright s$ for some $s \in S$. Then for some $m < \omega$, $s + n \cap s = \emptyset$

for all $n \geq m$, so $\sigma^n(\bar{b} \restriction s) = \bar{b} \restriction (s+n)$ satisfies that $\sigma^n(\bar{b} \restriction s) \mathop{\smile\!\!\!\!\!\!|}^* \bar{b} \restriction s$ and so as $\mathop{\smile\!\!\!\!\!\!|}^*$ implies non-splitting, σ is indeed strongly repulsive. $\square$

Corollary 3.13. *Suppose that M is a countable ω-categorical L-structure, and that $\mathop{\smile\!\!\!\!\!\!|}$ is a CIR on M. Then there is a strongly repulsive automorphism $\sigma \in \mathrm{Aut}(M)$.*

Proof. For all $n < \omega$ and $a \in M^n$, let $R_a \subseteq M^n$ be the orbit of a under $G = \mathrm{Aut}(M)$. Let $L' = \{R_a \mid a \in M^n, n < \omega\}$ and let M' be the L'-structure induced by M. Then M' has the same definable sets as M, has quantifier elimination and is ultrahomogeneous. Now, $\mathop{\smile\!\!\!\!\!\!|}$ is still a CIR on finite subsets of M'. Moreover, substructures of M' are subsets since L' is relational, so $\mathop{\smile\!\!\!\!\!\!|}$ is defined on finitely generated substructures. Thus we may apply Theorem 3.12 to get a strongly repulsive automorphism σ of M'. However, $\mathrm{Aut}(M) = \mathrm{Aut}(M')$ and σ is strongly repulsive as an automorphism of M' as well. $\square$

From Corollary 3.9 and Corollary 3.13 we get:

Corollary 3.14. *If M is a countable model of an ω-categorical theory which has a canonical independence relation then $G = \mathrm{Aut}(M)$ has a cyclically dense conjugacy class. In fact, there is some $f \in G$ such that the set of $g \in G$ for which [$\{f^n g f^{-n} \mid n \in \mathbb{Z}\}$ is dense] is comeager.*

Furthermore, by Proposition 3.5, the same is immediately true for G^n for any $n < \omega$.

3.1. Ramsey structures and a weakening of having a CIR.

Lemma 3.15. *Suppose that M is a countable ultrahomogeneous structure. Then (1) implies (2) implies (3) where:*

1. *M has a CIR defined on finitely generated substructures.*
2. *There are two models M_0, M_1 isomorphic to M and contained in M, such that $M_0 \mathop{\smile\!\!\!\!\!\!|}^{ns} M_1$ and $M_1 \mathop{\smile\!\!\!\!\!\!|}^{ns} M_0$.*
3. *There is a binary relation $\mathop{\smile\!\!\!\!\!\!|}$ on finite subsets of M that satisfies all the properties of Definition 3.10 but only over $\emptyset$. Namely it satisfies stationarity, extension to the right and the left (over $\emptyset$), monotonicity (over $\emptyset$) and existence (over $\emptyset$).*

Proof. (1) implies (2). Suppose that M has a CIR $\mathop{\smile\!\!\!\!\!\!|}$ defined on finitely generated substructures. Consider the tuple $\bar{b}$ constructed in the proof of Theorem 3.12. By condition (3) in that proof, both $\bar{b} \restriction [0, \infty)$ and $\bar{b} \restriction (-\infty, 0]$ are ultrahomogeneous and with the same age as M, thus isomorphic to M by Fact 2.1. By stationarity and monotonicity, $\bar{b} \restriction [0, \infty) \mathop{\smile\!\!\!\!\!\!|}^{ns} \bar{b} \restriction (-\infty, 0]$ and $\bar{b} \restriction (-\infty, 0] \mathop{\smile\!\!\!\!\!\!|}^{ns} \bar{b} \restriction [0, \infty)$.

(2) implies (3). For two finite sets $A, B \subseteq M$, let $A \mathop{\smile\!\!\!\!\!\!|} B$ iff there is some automorphism $\sigma \in \mathrm{Aut}(M)$ such that $\sigma(A) \subseteq M_0$ and $\sigma(B) \subseteq M_1$. Now, $\mathop{\smile\!\!\!\!\!\!|}$ is stationary: suppose that $a \mathop{\smile\!\!\!\!\!\!|} b$, $a' \mathop{\smile\!\!\!\!\!\!|} b'$, $a \equiv b$ and $a' \equiv b'$. We want to show that $ab \equiv a'b'$. Let $\sigma, \tau \in \mathrm{Aut}(M)$

be such that $\sigma(a), \tau(a') \subseteq M_0$ and $\sigma(b), \tau(b') \subseteq M_1$. Then $\sigma(a)\tau(a') \mathop{\smile\!\!\!\!\!\!\,|}^{ns} \sigma(b)\tau(b')$ and $\sigma(b)\tau(b') \mathop{\smile\!\!\!\!\!\!\,|}^{ns} \sigma(a)\tau(a')$, and hence $ab \equiv \sigma(ab) \equiv \sigma(a)\tau(b') \equiv \tau(a'b') \equiv a'b'$.

Next, $\mathop{\smile\!\!\!\!\!\!\,|}$ satisfies extension (on the right): suppose that $A \mathop{\smile\!\!\!\!\!\!\,|} B$ and d is a finite tuple of M, and we want to find some $d' \equiv_B d$ such that $A \mathop{\smile\!\!\!\!\!\!\,|} B \cup d'$. By definition, there is some σ with $\sigma(A) \subseteq M_0$ and $\sigma(B) \subseteq M_1$. As M_1 is ultrahomogeneous and B is finite, we can extend $\sigma \restriction \langle B \rangle$ to some $f : \langle Bd \rangle \to M_1$. As M is ultrahomogeneous, $d' = \sigma^{-1}(f(d)) \equiv_B d$ and σ witnesses that $A \mathop{\smile\!\!\!\!\!\!\,|} B \cup d'$. Extension on the left is shown in the same way.

Existence follows from the fact that $\mathrm{Age}(M_1) = \mathrm{Age}(M_0) = \mathrm{Age}(M)$. Monotonicity is clear.

$\square$

Remark 3.16. If in Lemma 3.15, if we had assumed that M was ω-categorical, in (3) we could define $\mathop{\smile\!\!\!\!\!\!\,|}$ on arbitrary subsets of M, even infinite. We would define $A \mathop{\smile\!\!\!\!\!\!\,|} B$ iff for every finite subsets $A' \subseteq A$ and $B' \subseteq B$, $A' \mathop{\smile\!\!\!\!\!\!\,|} B'$. One can then use König's Lemma and Ryll-Nardzewski to show that this has the extension property.

In addition, (3) would be equivalent to (2): enumerate M as $\bar{m} = \langle m \mid m \in M \rangle$ (i.e., the identity function). Let $\bar{x} = \langle x_m \mid m \in M \rangle$ and $\bar{y} = \langle y_m \mid m \in M \rangle$ be two disjoint sequences of variables. For a finite tuple $a = \langle m_i \mid i < n \rangle$ in M, write $x_a = \langle x_{m_i} \mid i < n \rangle$, and similarly define y_a. Let $\Gamma(\bar{x}, \bar{y})$ be the union of the sets $\Gamma^0_{a,b,c}(x_a, x_b, y_c)$ and $\Gamma^1_{a,b,c}(x_c, y_a, y_b)$ for all finite tuples a, b, c from M such that $a \equiv b$, where $\Gamma^0_{a,b,c}(x_a, x_b, y_c)$ says that x_a and x_b have the same type over y_c and similarly, $\Gamma^1_{a,b,c}(x_c, y_a, y_b)$ says that y_a and y_b have the same type over x_c. Let $\Sigma(\bar{x}, \bar{y})$ be $\Gamma(\bar{x}, \bar{y})$ and the assertions that both $\bar{x}$ and $\bar{y}$ satisfy the type $\mathrm{tp}(\bar{m}/\emptyset)$. Then by (3), Σ is consistent, so by ω-categoricity, we can realize it in M.

Proposition 3.17. *Suppose that M is a countable ω-categorical Ramsey structure. Then (2) from Lemma 3.15 holds.*

Proof. Let $G = \mathrm{Aut}(M)$ and let $\bar{m} = \langle m \mid m \in M \rangle$ (i.e., the identity function). Let $\bar{x} = \langle x_m \mid m \in M \rangle$ and $S_{\bar{m}}(M) = \{p(\bar{x}) \in S(M) \mid p \restriction \emptyset = \mathrm{tp}(\bar{m}/\emptyset)\}$, a compact Hausdorff space with the logic topology. Then G acts on $S_{\bar{m}}(M)$ by setting $\sigma * p = \{\sigma * \varphi \mid \varphi \in p\}$ where $\sigma * \varphi(\bar{x}, m) = \varphi(\bar{x}, \sigma(m))$. A fixed point of this action is just an invariant type over M. As G is extremely amenable (Fact 2.20), there is an invariant type which enumerates a model N such that that $N \mathop{\smile\!\!\!\!\!\!\,|}^{ns} M$. Now consider the space P of invariant types: $P = \{q \in S_{\bar{m}}(M) \mid G * q = q\}$. A type q is invariant iff for every formula $\varphi(\bar{x}, y)$ over $\emptyset$, if $m \equiv m'$ are from M then $\varphi(\bar{x}, m) \in q$ iff $\varphi(\bar{x}, m') \in q$. This is easily a closed condition, so P is compact, and we already know that it is nonempty. Now let G act on P by setting $\sigma \star q = \{\sigma \star \varphi \mid \varphi \in q\}$ where $\sigma \star \varphi(\bar{x}, m) = \varphi(\bar{x}_\sigma, m)$, where $\bar{x}_\sigma = \left\langle x_{\sigma(m)} \,\middle|\, m \in M \right\rangle$. Note that for all $p \in P$ and $\sigma \in G$, $\sigma \star p \restriction \emptyset = \mathrm{tp}\left(\left\langle \sigma^{-1}(m) \,\middle|\, m \in M \right\rangle / \emptyset\right) = \mathrm{tp}(\bar{m}/\emptyset)$, and that $\sigma \star p$ remains invariant. By extreme amenability, there is some $q \in P$ such that $\sigma \star q = q$ for all $\sigma \in G$. Let N' be a model enumerated by a realization of q, then $N' \mathop{\smile\hskip-0.9em ^{|}\,}^{ns} M$ and $M \mathop{\smile\hskip-0.9em ^{|}\,}^{ns} N'$.

By ω-categoricity, we may assume that these two models are contained in M. $\square$

Remark 3.18. By Fact 2.19, expanding an ultrahomogeneous Ramsey structure by finitely many constants gives an independence relation as in (3) from Lemma 3.15 over any finite set. However there is no reason that it would satisfy transitivity. Indeed, the lexicographically ordered dense tree (which is Ramsey, see Example 2.16) does not have a CIR. See Corollary 6.10 below.

4. Examples of theories with a canonical independence relation

There are many examples of countable ultrahomogeneous structures with a CIR. Here we will give some of them. We will define the relation $\mathop{\smile\hskip-0.9em ^{|}\,}$, but sometimes leave most of the details of checking that it satisfies the axioms to the reader. All the CIRs we define are defined on finitely generated substructures.

Example 4.1. The most trivial ultrahomogeneous structure is of course the structure with universe ω and no relations but equality. Its automorphism group is S_∞. For finite sets A, B, C define $A \mathop{\smile\hskip-0.9em ^{|}\,}_C B$ by $A \cap B \subseteq C$. This is a CIR.

Example 4.2. If T is stable and $\emptyset$ is a base (i.e., $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$ so that every type over $\emptyset$ has a unique non-forking extension), then $\mathop{\smile\hskip-0.9em ^{|}\,}^f$ (i.e., non-forking independence) is canonical.

Example 4.3. Let $(\mathcal{B}, <, \wedge, \vee, 0, 1, \square^c)$ be the atomless Boolean algebra. For finite sets A, B, C, define $A \mathop{\smile\hskip-0.9em ^{|}\,}_C B$ iff $\langle C \rangle = \langle AC \rangle \cap \langle BC \rangle$ and for every atom $a \in \langle AC \rangle$ and every atom $b \in \langle BC \rangle$, if there is an atom $c \in \langle C \rangle$ such that $a, b \leq c$ then $a \wedge b \neq 0$. Let us show that $\mathop{\smile\hskip-0.9em ^{|}\,}$ is a (symmetric) CIR.

Stationarity over $\emptyset$: $A \mathop{\smile\hskip-0.9em ^{|}\,} B$ says that the atoms in $\langle AB \rangle$ are in bijection with (atoms of A) $\times$ (atoms of B). Thus, if $A' \mathop{\smile\hskip-0.9em ^{|}\,} B'$ and $f : \langle A \rangle \to \langle A' \rangle$, $g : \langle B \rangle \to \langle B' \rangle$ are isomorphisms, then $h : \langle AB \rangle \to \langle A'B' \rangle$ taking an atom $a \wedge b$ to $f(a) \wedge g(b)$ is an isomorphism. This easily implies stationarity.

Transitivity: suppose that $A \mathop{\smile\hskip-0.9em ^{|}\,}_{CD} B$ and $D \mathop{\smile\hskip-0.9em ^{|}\,}_C B$, and we want to show that $AD \mathop{\smile\hskip-0.9em ^{|}\,}_C B$. Suppose that $a \in \langle ACD \rangle$, $b \in \langle BC \rangle$ are atoms, and $c \in \langle C \rangle$ is an atom such that $a, b \leq c$. Let $d \in \langle DC \rangle$ be an atom such that $a \leq d \leq c$. Then $d \wedge b \neq 0$. Let $b' \leq d \wedge b$ be an atom of $\langle BCD \rangle$, so that $a, b' \leq d$. Hence $a \wedge b' \neq 0$ and thus $a \wedge b \neq 0$. Transitivity to the right follows by symmetry.

Extension: suppose that $A \mathop{\smile\hskip-0.9em ^{|}\,}_C B$ and d is given. We want to find $d' \equiv_{BC} d$ such that $A \mathop{\smile\hskip-0.9em ^{|}\,}_C Bd$. We may assume that $d \notin \langle BC \rangle$. An atom in $\langle BCd \rangle$ has the form $d \wedge b$ or $d^c \wedge b$ for some atom b of $\langle BC \rangle$. The type $\mathrm{tp}(d/BC)$ is determined by knowing which of these terms $d \wedge b, d^c \wedge b$ is nonzero (for $b \in \langle BC \rangle$ an atom). As $\mathcal{B}$ is atomless and $A \mathop{\smile\hskip-0.9em ^{|}\,}_C B$, we can find some d' such that for every

a, b, c atoms in $\langle AC \rangle, \langle BC \rangle, \langle C \rangle$ respectively such that $a, b \leq c$, if $d \wedge b \neq 0$ then $a \wedge d' \wedge b \neq 0$ (else $d' \wedge b = 0$) and if $d^c \wedge b \neq 0$ then $a \wedge (d')^c \wedge b \neq 0$ (else $(d')^c \wedge b = 0$). In addition, we ask that if $d \wedge b, d^c \wedge b \neq 0$ then both $a \wedge d' \wedge b$ and $a \wedge (d')^c \wedge b$ are not in $\langle ABC \rangle$. It now follows that $\langle AC \rangle \cap \langle BCd' \rangle = \langle C \rangle$: suppose that $e \in \langle AC \rangle \cap \langle BCd' \rangle$. Then as $e \in \langle BCd' \rangle$, it can be written as $b_0 \vee (b_1 \wedge d') \vee \left(b_2 \wedge (d')^c\right)$ where $b_0, b_1, b_2 \in \langle BC \rangle$ are pairwise disjoint and for every atom $b' \leq b_1 \vee b_2$ from $\langle BC \rangle$, $d' \wedge b', (d')^c \wedge b' \neq 0$. If both $b_1, b_2 = 0$, then $e \in \langle BC \rangle \cap \langle AC \rangle$ so $e \in \langle C \rangle$. If $b_1 \neq 0$, let $b_1' \leq b_1$ be an atom of $\langle BC \rangle$. So $e \wedge b_1' \in \langle ABC \rangle$ (because $e \in \langle AC \rangle$) and has the form $b_1' \wedge d'$. Let $a \in \langle AC \rangle$ and $c \in \langle C \rangle$ be atoms such that $a, b_1' \leq c$. Then $e \wedge b_1' \wedge a \in \langle ABC \rangle$ and has the form $a \wedge b_1' \wedge d'$ which is not in $\langle ABC \rangle$ by construction, contradiction. Similarly $b_2 = 0$ and we are done.

Existence and monotonicity are clear.

Example 4.4. Let (M, R) be the random tournament (a tournament is a complete directed graph such that for all x, y, it cannot be that both $R(x, y)$ and $R(y, x)$, and the random tournament is the Fraïssé limit of the class of finite tournaments). Given finite sets A, B, C, write $A \downarrow_C B$ iff $A \cap B = C$ and if $a \in A \backslash C$, $b \in B \backslash C$ then $R(a, b)$. This easily satisfies all the requirements.

Remark 4.5. The following definition is from [KM17, after Theorem 5.13]. Let K be a class of finite L-structures where L is a relational language. Then K has the *strong$^+$ amalgamation property* if for all $A, B, C \in K$ with $A \subseteq B, C$ there is $D \in K$ such that $D = C' \cup B'$ with $C' \cong_A C$, $B' \cong_A B$ and for every n-ary relation R and every $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ from $D \backslash A$ which intersect both C' and B', if $[x_i \in B'$ iff $y_i \in B'$ for all $1 \leq i \leq n]$, then $R^D(x_1, \ldots, x_n)$ iff $R^D(y_1, \ldots, y_n)$ (where R^D is the interpretation of R in D). For example, the random tournament satisfies this property. In [KM17, after Theorem 5.14] it is proved that if M is ultrahomogeneous and $\mathrm{Age}(M)$ has the strong$^+$ amalgamation property, then $\mathrm{Aut}(M)$ has a cyclically dense conjugacy class. They prove it using a condition they denote by (Δ_n), see there, Theorem 5.12. We do not know if this condition implies the existence of a CIR.

4.1. **Free amalgamation classes.** In [TZ13, Mül16, Con17] there is an axiomatic framework for defining an abstract ternary relation close to our CIR. More precisely, in [Mül16, Definition 3.1] and [TZ13, Definition 2.1], the notion of a stationary independence relation (SIR) is introduced (in [Mül16, Definition 3.1] for finitely generated structures and in [TZ13, Definition 2.1] for sets in general). A similar notion is defined in [Con17, Definition 2.1], with more axioms. In any case, all these notions imply ours, except perhaps that stationarity over $\emptyset$ becomes stationarity over $\mathrm{acl}(\emptyset)$, so that this becomes a CIR in the expansion $(M, \mathrm{acl}(\emptyset))$ (note that our extension follows from full stationarity and their version of existence).

Thus, we can apply our results to the examples studied there. In particular, we get the following examples.

Example 4.6. The rational Urysohn space $\mathbb{QU}$ is the Fraïssé limit of the class of finite metric spaces with rational distances. Pick a point $q \in \mathbb{QU}$, and consider the structure $(\mathbb{QU}, q)$ where we add a constant for q. In [TZ12], it is proved that the relation $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ which holds for finite A, B, C iff for every $a \in AC$, $b \in BC$, $d(a,b) = \min\{d(a,c) + d(c,b) \mid c \in C\}$ is a CIR in $(\mathbb{QU}, q)$.

In all examples given by Conant [Con17, Example 3.2] which we list now, $\operatorname{acl}(\emptyset) = \emptyset$, so we actually get a CIR in the structure (i.e., no need to take an expansion) by [Con17, Proposition 3.4].

Example 4.7. Fraïssé limits with free amalgamation: suppose that L is a relational language and K is an essentially countable (see above Fact 2.1) class of finite L-structures, such that if $A, B, C \in K$ and $A \subseteq C, B$, then the free amalgam of A, B, C is in K (i.e., a structure $D = C' \cup B'$ with $C' \cong_A C$, $B' \cong_A B$, $B' \cap C' \subseteq A$ and for every tuple $a \in D$ in the length of some relation $R \in L$, if $R(a)$ then $a \in C'$ or $a \in B'$) (here we also include the case $A = \emptyset$). Let M be the Fraïssé limit of K, and define $B \mathop{\smile\!\!\!\!\!\!\mid}_A C$ iff ABC is the free amalgam of A, AB, AC. It is easy to see that in this case $\mathop{\smile\!\!\!\!\!\!\mid}$ is a CIR (this is also proved, for finite languages, in [Con17, Proposition 3.4]).

This class of examples contain e.g., the random graph, the universal K_n-free graph (the Henson graph), and their hypergraph analogs.

Example 4.8. Let $L = \{P_n \mid n < \omega\}$ and let K be the class of finite L-structures in which $P_n(x_0, \ldots, x_{n-1})$ implies that $x_i \neq x_j$ for $i \neq j$. Then K is essentially countable and is a free amalgamation class. Now recall the example N described in Remark 3.2, with infinitely many independent equivalence relations with two classes. Then M is N expanded by naming the classes. In other words, $\operatorname{Aut}(M) = \operatorname{Aut}(N)^0$.

Example 4.9. In [CSS99, Section 10] the authors describe a generic $K_n + K_3$-free graph, where $K_n + K_3$ is the free amalgam of the complete graph on n vertices and a triangle over a single vertex. This structure is $\aleph_0$-categorical, with $\operatorname{acl}(\emptyset) = \emptyset$. By [Con17, Example 3.2 (2), Proposition 3.4] there is a CIR on this graph.

Example 4.10. ω-categorical Hrushovski constructions. Let L be a finite relational language, and let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a "control function". According to [Eva02, Theorem 3.5], there is an ω-categorical generic Hrushovski construction M_f for a "free amalgamation class" K_f if f satisfies certain conditions. It follows from [Con17, Example 3.2 (3), Proposition 3.4] that given extra conditions on the algebraic closure, M_f admits a CIR. See more details in [Con17, Eva02].

4.2. **Ultrahomogenous partial orders.** In [GGS17] the authors prove that the automorphism group of every ultrahomogeneous poset (partially order set) is topologically 2-generated. They also characterize when they have a cyclically dense conjugacy class. We can find such a conjugacy class by finding a CIR whenever possible. We should remark that they prove more on the automorphism groups of those structures. By [Sch79] there are four types of ultrahomogeneous posets.

Fact 4.11. [Sch79] *Suppose that $(H,<)$ is an ultrahomogeneous poset. Then H is isomorphic to one of the following:*

(1) *The random poset: the Fraïssé limit of the class of finite partial orders.*
(2) *The orders $\mathcal{A}_n$ for $1 \leq n \leq \omega$: $(n,<)$ where $<$ is trivial i.e., empty.*
(3) *The orders $\mathcal{B}_n$ for $1 \leq n \leq \omega$: $(n \times \mathbb{Q},<)$ where $(k,q)<(m,p)$ iff $k=m$ and $p<q$.*
(4) *The orders $\mathcal{C}_n$ for $1 \leq n \leq \omega$: $(n \times \mathbb{Q},<)$ where $(k,q)<(m,p)$ iff $q<p$.*

Note that the orders $\mathcal{A}_n$ have S_n as their automorphism group, and thus for n finite cannot have a dense conjugacy class. For $n=\omega$, this is Example 4.1.

Also, the orders $\mathcal{B}_n$ for $1<n<\omega$ cannot have a dense conjugacy class by Remark 3.2: S_n is a quotient of the automorphism group (define $a\,E\,b$ iff a and b are comparable, and note that there are n equivalence classes, every permutation of which is induced by an automorphism).

4.2.1. *The random poset.* Suppose that $(\mathcal{D},\leq)$ is the random partial order. For finite sets A,B,C define $A \mathop{\smile\!\!\!\!\!\!\;\,|}_C B$ iff $A \cap B \subseteq C$ and if $a \in A, b \in B$ then a is comparable with b iff for some $c \in C$, $a \leq c \leq b$ or $b \leq c \leq a$. Then $\mathop{\smile\!\!\!\!\!\!\;\,|}$ is a (symmetric) CIR. We will show only transitivity and extension, and leave the rest to the reader. Suppose that $A \mathop{\smile\!\!\!\!\!\!\;\,|}_{CD} B$ and $D \mathop{\smile\!\!\!\!\!\!\;\,|}_C B$. Given $a \in A, b \in B$, such that $a \leq b$, there must be some $d \in CD$ such that $a \leq d \leq b$. Hence there must be some $c \in C$ with $d \leq c \leq b$. Together we are done. Extension: suppose that $A \mathop{\smile\!\!\!\!\!\!\;\,|}_C B$ and we are given d. Assume $d \notin BC$ (otherwise we are done). Then let $d' \equiv_{BC} d$ be such that $d' \notin ABC$ and for all $a \in A$, $a \leq d'$ iff for some $c \in C$, $a \leq c \leq d$, and similarly define when $d' \leq a$. Now, $(ABCd',\leq)$ is a poset since $A \mathop{\smile\!\!\!\!\!\!\;\,|}_C B$.

4.2.2. *The orders $\mathcal{B}_1$ and $\mathcal{B}_\omega$.*

Example 4.12. For $(\mathbb{Q},<)$ (which is $\mathcal{B}_1$), for every finite $A,B,C \subseteq \mathbb{Q}$, we let $A \mathop{\smile\!\!\!\!\!\!\;\,|}_C B$ if $A \cap B \subseteq C$ and for all $a \in A \backslash C$ and $b \in B \backslash C$ such that $a \equiv_C b$, $a<b$. Then $\mathop{\smile\!\!\!\!\!\!\;\,|}$ is a CIR. We prove only transitivity and leave the rest to the reader.

Suppose that $A \mathop{\smile\!\!\!\!\!\!\;\,|}_{DC} B$ and $D \mathop{\smile\!\!\!\!\!\!\;\,|}_C B$. We have to show that $AD \mathop{\smile\!\!\!\!\!\!\;\,|}_C B$, which amounts to showing that $A \mathop{\smile\!\!\!\!\!\!\;\,|}_C B$. Fix $a \in A \backslash C$ and $b \in B \backslash C$ such that $a \equiv_C b$. We have to show that $a<b$. Note that $b \notin D$. If $a \in D$ then this is true by our assumption. Otherwise, $a \notin CD$. If $a \equiv_{CD} b$ then we are done. Otherwise, a,b have different cuts over CD. But since they realize the same cut over C, it follows that there is some $d \in D \backslash C$ such that either $a<d<b$ or $b<d<a$. The former would imply what we want, so assume that $b<d<a$. But then $b \equiv_C d$ so $d<b$ — a contradiction. The other direction of transitivity is proved similarly.

Example 4.13. Consider $\mathcal{B}_\omega$. Then each equivalence class of the relation E of being comparable is a DLO, and thus by Example 4.12, for each $n<\omega$, there is a CIR $\mathop{\smile\!\!\!\!\!\!|}^F$ defined as in Example 4.12 for each E-class F. For $(\mathcal{B}_\omega,<)$ and finite sets A,B,C, define $A \mathop{\smile\!\!\!\!\!\!|}_C B$ iff $A\cap B\subseteq C$, $A/E\cap B/E\subseteq C/E$(if $a\in A$, $b\in B$ and $a\mathrel{E} b$ then there is some $c\in C$ such that $a\mathrel{E} c$) and for every E-class F, $A\cap F \mathop{\smile\!\!\!\!\!\!|}^F_{C\cap F} B\cap F$. This is easily seen to be a CIR. Note that we need infinitely many classes for extension.

4.2.3. *The orders $\mathcal{C}_n$ for $1\le n\le\omega$.* In $\mathcal{C}_n$ we have an equivalence relation E, defined by $a\mathrel{E} b$ iff a and b are incomparable (they have the same second coordinate). Then $\mathcal{C}_n/E\models DLO$, so we have a CIR $\mathop{\smile\!\!\!\!\!\!|}^E$ defined on it by Example 4.12. For finite $A,B,C\subseteq\mathcal{C}_n$, define $A \mathop{\smile\!\!\!\!\!\!|}_C B$ iff $A/E \mathop{\smile\!\!\!\!\!\!|}^E_{C/E} B/E$. This trivially satisfies all the properties.

4.3. **Ultrahomogeneous graphs.** In [JM17], the authors prove that for every ultrahomogeneous graph $\Gamma=(V,E)$, $\mathrm{Aut}(\Gamma)$ is topologically 2-generated. Similarly to the poset case, we can recover some results by finding a CIR whenever possible. By [LW80] we have the following classification of ultrahomogeneous graphs. Recall that for a graph (V,E), its *dual* is (V,E') where $E'=[V]^2\setminus E$.

Fact 4.14. [LW80] *Any countable ultrahomogeneous graph Γ is isomorphic to one of the following graphs, or its dual.*

(1) *The random graph.*
(2) *For $n\ge 3$, the Henson graph, i.e., the K_n-free universal graph (the Fraïssé limit of the class of K_n-free finite graphs).*
(3) *For any $1\le n\le\omega$, the graph ωK_n consisting of a disjoint union of countably many copies of K_n.*
(4) *For any $2\le n<\omega$, the graph nK_ω consisting of a disjoint union of n copies of K_ω (the complete graph on ω).*

Note that the dual of a graph has the same automorphism group, so we can ignore the duals.

We already saw in Example 4.7 that both the random graph and the Henson graph have a CIR. The graphs nK_ω for $n<\omega$ cannot have a a dense conjugacy class by Remark 3.2 as in the case of the posets $\mathcal{B}_n$ described above. However, ωK_n for $1\le n\le\omega$ has a CIR, just like the cases $\mathcal{C}_n$ above.

4.4. **A mix of two Fraïsé limits with CIRs.** Suppose we are in the situation of Section 2.2: we have two amalgamation classes K_1,K_2 with all the properties listed there. Let M_1,M_2 be the Fraïsé limits of K_1,K_2 respectively, and let M be the Fraïsé limit of K, the class of finite $L_1\cup L_2$-structures A such that $A\restriction L_1\in K_1$ and $A\restriction L_2\in K_2$. Add the extra assumption that $L_1\cap L_2=\emptyset$. Suppose that $\mathop{\smile\!\!\!\!\!\!|}^1,\mathop{\smile\!\!\!\!\!\!|}^2$ are CIRs on M_1,M_2 respectively. By Proposition 2.4, we may assume that $M_1 = M \upharpoonright L_1$ and $M_2 = M \upharpoonright L_2$. For finite subsets of M, define $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ iff $A \mathop{\smile\!\!\!\!\!\!\mid}_C^1 B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_C^2 B$.

Proposition 4.15. *The relation $\mathop{\smile\!\!\!\!\!\!\mid}$ is a CIR.*

Proof. Stationarity follows from the fact that by quantifier elimination, for any finite tuples a, a' from M, if $a \equiv a'$ in L_1 and in L_2, then $a \equiv a'$ in $L_1 \cup L_2$.

Extension: suppose that $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$, and we are given $d \in M$. Let $d_1 \in M_1$ be such that $d_1 \equiv_{BC} d$ in L_1 and $A \mathop{\smile\!\!\!\!\!\!\mid}_C^1 Bd_1$. Similarly find d_2 for $\mathop{\smile\!\!\!\!\!\!\mid}^2$ and L_2. Consider the finite structure D with universe $ABCd$ where the $L_1 \cup L_2$-structure on ABC is as in M, and such that its restriction to L_1, L_2 is $ABCd_1$, $ABCd_2$, respectively. This structure exists since $L_1 \cap L_2 = \emptyset$ and by the assumptions of Section 2.2, both languages are relational. Thus, $D \in K$, so it has an isomorphic copy $D' \subseteq M$ containing copies A', B', C', d' of A, B, C, d. As M is ultrahomogeneous, we can apply an automorphism σ mapping $A'B'C'$ to ABC, so that $A \mathop{\smile\!\!\!\!\!\!\mid}_C B\sigma(d')$, and $\sigma(d') \equiv_{BC} d$.

The other properties are easy to check. □

Example 4.16. The ordered random graph $M = (V, R, <)$. It is the Fraïsé limit of the class of finite linearly ordered graphs in the language $\{<, R\}$. It easily satisfies all our assumptions with $L_1 = \{<\}$, K_1 the class of finite linear orders and $L_2 = \{R\}$, K_2 the class of finite graphs. It has a CIR as both $(M, <)$ (which is a DLO) and (M, R) (the random graph) have CIRs by the two previous subsections. Similarly we may define the random ordered hypergraph, and it too has a CIR.

4.5. **Trees.** The theory T_{dt} (see Example 2.3) does not admit a canonical independence relation. We shall give a precise (and stronger) argument for this below in Corollary 6.10, but it is easy to see that a natural candidate fails. Namely, one can try to define $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ in such a way that if $C = \emptyset$ and a, b are singletons then $a \mathop{\smile\!\!\!\!\!\!\mid} b$ iff $a \wedge b < a, b$, and for a, b, c such that $c \mathop{\smile\!\!\!\!\!\!\mid} b$, then $a \mathop{\smile\!\!\!\!\!\!\mid}_c b$ iff $a \wedge c < c \wedge b$. But then $a \wedge c \mathop{\smile\!\!\!\!\!\!\mid}_c b$, $c \mathop{\smile\!\!\!\!\!\!\mid} b$ but $a \wedge c \not\mathop{\smile\!\!\!\!\!\!\mid} b$, so transitivity fails.

However, we can expand it in such a way that it does. We give two such expansions.

Example 4.17. Let $L_{dt}^B = \{<, P, f, \wedge\}$ where P is a unary predicate and f is a unary function symbol, and let T_{dt}^B be the model completion of the universal L_{dt}^B-theory of trees where P is a downwards closed linearly ordered subset and $f(x)$ is the maximal element in P which is $\leq x$. In other words, T_{dt}^B is the theory of the Fraïsé limit of the class of finite L_{dt}^B-structures M where $M \upharpoonright \{\wedge, <\}$ is a tree with a meet function, P^M is linearly ordered and downwards closed and $f(x) = \max\{y \leq x \mid y \in P\}$ (note that this class has JEP and AP). Then T_{dt}^B is the theory of dense trees with a predicate for a branch (a maximal chain), it has quantifier elimination and is ω-categorical. Let us see why P is a maximal chain in every model $M \models T_{dt}^B$. Of course it is downwards closed by definition, so if $a \in M$ is comparable with P but $a \notin P$, then $a > P$. As T_{dt}^B

is model-complete, M is existentially closed (see Fact 2.1) so there is some $b \in P$ (from M) such that $f(a) < b$. Thus, $a > b > f(a)$ which is a contradiction to the definition of f.

For three sets A, B, C, let $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ iff $\langle AC\rangle \cap \langle BC\rangle \subseteq \langle C\rangle$ and for all $a \in \langle AC\rangle$ with $f(a) \notin \langle C\rangle$ and $b \in \langle BC\rangle$ with $f(b) \notin \langle C\rangle$ such that $f(a) \equiv_C f(b)$ (which is the same as $f(a) \equiv_{f(C)} f(b)$), $f(a) < f(b)$. Then $\mathop{\smile\!\!\!\!\!\!\mid}$ is canonical. The only nontrivial axioms to check are stationarity over $\emptyset$, extension and transitivity.

Suppose that $A \mathop{\smile\!\!\!\!\!\!\mid} B$. This just says that that B is placed above A with respect to the branch P (i.e., $f(a) < f(b)$ for all $a \in A, b \in B$). So $\mathop{\smile\!\!\!\!\!\!\mid}$ is stationary by quantifier elimination.

Extension: suppose that $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and we are given a single element d, which we may assume is not in $\langle BC\rangle$ and even that $f(d) \notin \langle BC\rangle$. First find some d'' such that $d'' \equiv_{BC} f(d)$ and d'' is greater than every $f(a)$ such that $a \in \langle AC\rangle$ and $f(a) \equiv_C f(d)$. Then find d' such that $d' \equiv_{BC} d$ and $f(d') = d''$ (and $\langle AC\rangle \cap \langle BCd'\rangle = \langle C\rangle$).

Transitivity: suppose that $A \mathop{\smile\!\!\!\!\!\!\mid}_{DC} B$ and $D \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and we have to show that $AD \mathop{\smile\!\!\!\!\!\!\mid}_C B$. Suppose that $a \in \langle ADC\rangle, f(a) \notin \langle C\rangle$ and $b \in \langle BC\rangle, f(b) \notin \langle C\rangle$ are such that $f(a) \equiv_C f(b)$ but $f(b) \le f(a)$. Then there must be some $d \in \langle DC\rangle$ such that $f(b) \le d \le f(a)$, as otherwise $f(a) \equiv_{CD} f(b)$. But since $d \mathop{\smile\!\!\!\!\!\!\mid}_C B$, $f(b) < d$, which implies that $f(b)$ and d do not have the same type over C, so there must be some $c \in C$ between them, and in particular, it contradicts our assumption that $f(a) \equiv_C f(b)$. The other direction of transitivity is proved similarly.

Example 4.18. Let $L_{dt}^p = \{<, p, \wedge\}$ where p is a new constant. Let T_{dt}^p be the unique completion of T_{dt} to L_{dt}^p. Let $M \models T_{dt}^p$ be the unique countable model. To simplify notation, we identify p with p^M. For three sets A, B, C, we let $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ iff $\langle AB\rangle \cap \langle BC\rangle \subseteq \langle C\rangle$ and:

(1) For all $a \in \langle AC\rangle$ with $a \wedge p \notin \langle C\rangle$ and $b \in \langle BC\rangle$ with $b \wedge p \notin \langle C\rangle$ such that $a \wedge p \equiv_C b \wedge p$, $a \wedge p < b \wedge p$.
(2) For all $a \in \langle AC\rangle$ such that $a > p$ with no $c \in \langle C\rangle$ such that $a \wedge c > p$, and all $b \in \langle BC\rangle$ with $b > p$ and no $c \in \langle C\rangle$ such that $b \wedge c > p$, $a \wedge b = p$.

Then $\mathop{\smile\!\!\!\!\!\!\mid}$ is canonical. The only nontrivial axioms to check are stationarity over $\emptyset$, extension and transitivity.

It is stationary over $\emptyset$ by elimination of quantifiers, since $A \mathop{\smile\!\!\!\!\!\!\mid} B$ iff $A' = \{a \in A \mid a \wedge p < p\}$ is placed below $B' = \{b \in B \mid b \wedge p < p\}$ with respect to the points below p while $A'' = \{a \in A \mid a \ge p\}$ and $B'' = \{b \in B \mid b \ge p\}$ are placed independently above p.

Extension: suppose that $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ and we are given d such that $d \notin \langle BC\rangle$. First assume that $d \wedge p < p$. If $d \wedge p \notin \langle BC\rangle$, similarly to Example 4.17, first find some d'' such that $d'' \equiv_{BC} d \wedge p$ and $d'' > a \wedge p$ for all $a \in \langle AC\rangle$ with $a \wedge p \equiv_C d \wedge p$. Then find d' such that $d' \equiv_{BC} d$ with $d' \wedge p \equiv d''$ (and $\langle BCd'\rangle \cap \langle AC\rangle = \langle C\rangle$). Now assume that $d > p$. If there is some $b \in \langle BC\rangle$ with

$b \wedge d > p$, any $d' \equiv_{BC} d$ such that $\langle BCd' \rangle \cap \langle AC \rangle = \langle C \rangle$ will work. Otherwise find some $d' \equiv_{BC} d$ such that $d' \wedge a = p$ for all $a \in \langle AC \rangle$ with $a > p$.

Transitivity: suppose that $A \mathop{\smile\!\!\!\!\!\mid}_{DC} B$ and $D \mathop{\smile\!\!\!\!\!\mid}_{C} B$ and we have to show that $AD \mathop{\smile\!\!\!\!\!\mid}_{C} B$. If we are in case (1) of the definition, ($a \in \langle ADC \rangle$, $a \wedge p \notin \langle C \rangle$, etc.) then we proceed exactly as in Example 4.17. Otherwise, suppose that $a \in \langle ADC \rangle$, $b \in \langle BC \rangle$ are as in case (2). If there is some $d \in \langle CD \rangle$ with $a \wedge d > p$, then for no $c \in \langle C \rangle$ is it the case that $c \wedge d > p$ (otherwise $a \wedge c > p$). Thus $d \wedge b = p$ because $D \mathop{\smile\!\!\!\!\!\mid}_{C} B$ hence $a \wedge b = p$ as required. If there is no such d then $a \wedge b = p$ because $A \mathop{\smile\!\!\!\!\!\mid}_{DC} B$.

Trees also satisfy the following interesting phenomenon.

Proposition 4.19. *If M is a dense tree as in Example 2.3 (i.e., the model companion of the theory of trees in $\{<, \wedge\}$) then for every $\sigma \in \mathrm{Aut}(M)$ which does not have any fixed points, there is a branch $B \subseteq M$ such that $\sigma(B) = B$.*

Proof. Let B be a maximal linearly ordered set such that $\sigma(B) = B$ (which exists by Zorn's lemma). We will show that B is a branch. Note that if $x \in B$ and $y < x$, then $B \cup \{\sigma^n(y) \mid n \in \mathbb{Z}\}$ is still a chain: given any $z \in B$ and any $n \in \mathbb{Z}$, $\sigma^n(x), z$ are comparable and $\sigma^n(y) < \sigma^n(x)$ it follows that $\sigma^n(y)$ and z are comparable (if $z \leq \sigma^n(x)$ then both $\sigma^n(y), z \leq \sigma^n(x)$, so they are comparable by the tree axioms, and if $\sigma^n(x) < z$, then $\sigma^n(y) < z$), and for any $n, m \in \mathbb{Z}$, $\sigma^n(y), \sigma^m(y)$ are comparable since $\sigma^n(x)$ and $\sigma^m(x)$ are (if $\sigma^n(x) \leq \sigma^m(x)$ then both $\sigma^n(y), \sigma^m(y) \leq \sigma^m(x)$ so they are comparable by the tree axioms). Hence B is downwards closed.

Now, as σ has no fixed points, B cannot have a maximum (which would have to be a fixed point). Also, if $a \geq B$ and $\sigma(a) \geq a$ or $\sigma(a) \leq a$ then $B \cup \{\sigma^n(a) \mid n \in \mathbb{Z}\}$ is still a chain (since $\sigma^n(a) \geq \sigma^n(B) = B$ for all $n \in \mathbb{Z}$), so $a \in B$.

If B is not a branch (in particular, if $B = \emptyset$, which we haven't ruled out yet), there is some $a \in M$ such that $B < a$. Let $b = \sigma(a) \neq a$ (and by the above, b, a are not comparable), so $B < b$. Hence $B \leq (a \wedge b) < a, b$. Now, $\sigma(a \wedge b) < \sigma(a) = b$, so $a \wedge b$ and $\sigma(a \wedge b)$ are comparable. The previous paragraph implies that $a \wedge b \in B$. But then B has a maximum — contradiction. □

5. Having finite topological rank

In this section we will find some criteria that ensure that G has finite topological rank.

5.1. ω-categorical stable theories.

Proposition 5.1. *If T is stable ω-categorical, $M \models T$ is countable and $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ is finite, then $\mathrm{Aut}(M)$ has finite topological rank.*

Proof. Without loss of generality, $M = M^{\mathrm{eq}}$ (if $S \subseteq \mathrm{Aut}(M^{\mathrm{eq}})$ generates a dense subgroup, then $S \restriction M = \{f \restriction M \mid f \in S\}$ generates a dense subgroup of $\mathrm{Aut}(M)$). Let $N = M_{\mathrm{acl}(\emptyset)}$ (i.e., name the elements in $\mathrm{acl}(\emptyset)$). Then N is ω-categorical by Propsition 2.13. Then in N, $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$, so by Example 4.2, there is a canonical independence relation in N, so $G^0 = \mathrm{Aut}(N)$ is topologically 2-generated by Corollary 3.14, say by $\{f_1, f_2\}$. Now, $\mathrm{Aut}(M)/G^0$ is finite by assumption, so let $S \subseteq \mathrm{Aut}(M)$ be a finite set of representatives. Then $S \cup \{f_1, f_2\}$ generates a dense subgroup $\mathrm{Aut}(M)$: given two finite tuples $\bar{a}, \bar{b}$ from M such that $\bar{a} \equiv \bar{b}$, there is an automorphism $\sigma \in \mathrm{Aut}(M)$ such that $\sigma(\bar{a}) = \bar{b}$. Also, there is some $f \in S$ such that $f^{-1}\sigma \in \mathrm{Aut}(N)$. Hence for some g in the group generated by $\{f_1, f_2\}$, $g(\bar{a}) = f^{-1}\sigma(\bar{a}) = f^{-1}(\bar{b})$, so $fg(\bar{a}) = \bar{b}$. □

The following fact implies immediately the next result.

Fact 5.2. [EH93, Lemma 3.1] *If T is ω-categorical and ω-stable and $M \models T$ is countable, then* $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ *is finite.*

Corollary 5.3. *If T is ω-stable and ω-categorical and $M \models T$ is countable, then* $\mathrm{Aut}(M)$ *has finite topological rank.*

5.2. **Reducing finite topological rank to expansions.** Suppose that M is countable let $G = \mathrm{Aut}(M)$. We now want to explore the idea that perhaps by expanding M (i.e., moving to a subgroup), we can show that the topological rank of G is small by showing that the rank of the automorphism group of the expansion is. Suppose that $H \leq G$. If (G, H) has a compact quotient (see Definition 2.9), then we cannot hope to deduce anything. For example, by Proposition 2.13 we have that G^0 acts oligomorphically on M and it can be that G^0 has a cyclically dense conjugacy class (so topological rank 2) while $G/G^0 = (\mathbb{Z}/2\mathbb{Z})^{\omega}$ (so G is not topologically finitely generated) — this happens in the example described in in Remark 3.2, see Example 4.8. Indeed, we will see that (G, H) having a compact quotient is the only obstruction.

5.2.1. *ω-categorical structures with finitely many reducts.*

Theorem 5.4. *Suppose that $H \leq G$ is closed and that (G, H) has no compact quotients. If there are only finitely many closed groups between G and H then there is some $g \in G$ such that $H \cup \{g\}$ topologically generate G.*

Remark 5.5. The condition of having finitely many closed groups in the theorem holds when for instance M is a reduct of an ω-categorical structure M' where $H = \mathrm{Aut}(M')$, and M' has only finitely many reducts up to bi-definability.

Proof. Let $\{H_i \mid i < n\}$ be the family of closed proper subgroups of G containing H (which is finite by assumption). If $[G : H_i] < \infty$ for some $i < n$, then there would be a closed normal proper subgroup $N_i \trianglelefteq G$ of finite index such that $N_i \leq H_i$ (in general, if $H' \leq G$ is closed of finite index, then there is a closed normal subgroup $N \leq H'$, $N \trianglelefteq G$ such that $[G : N] < \infty$. In fact, $N = \bigcap \left\{gH'g^{-1} \,\middle|\, g \in G\right\}$ and this intersection is finite as it is the orbit of H' under the action of G on conjugates of H' and its stabilizer contains H'). But then $N_i H = G$ by assumption and Proposition 2.10, so $G = N_i H \subseteq H_i H = H_i$ contradicting the fact that H_i was a proper subgroup.

By a theorem of Neumann [Neu54, Lemma 4.1], there is some $g \in G \setminus \bigcup \{H_i \,|\, i < n\}$. If $G \neq \mathrm{cl}(\langle H \cup \{g\}\rangle)$ (the topological closure of the group generated by $H \cup \{g\}$), then $\mathrm{cl}(\langle H \cup \{g\}\rangle)$ is one of the groups H_i, contradicting the choice of g. □

Corollary 5.6. *If G and H are as in Theorem 5.4 and H has finite topological rank then so does G.*

By Example 2.12, in the ω-categorical context we get that if $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$ in M and M' is an expansion having finitely many reducts, then we can apply Corollary 5.6. This is the case, for instance, when M' is $(\mathbb{Q}, <)$ (see [JZ08]). By Lemma [JZ08, Lemma 2.10], an example of such a reduct of DLO is given by the countable dense circular order, which is the structure with universe $\mathbb{Q}$, and a ternary relation $C(x, y, z)$ given by $C(x, y, z) \Leftrightarrow x < y < z \vee y < z < x \vee z < x < y$.

Corollary 5.7. $\mathrm{Aut}(\mathbb{Q}, C)$ *has topological rank* ≤ 3*, but* $(\mathbb{Q}, C)$ *has no CIR.*

Proof. We only have to show that it has no CIR. By Lemma 3.15, if there was a CIR, then in particular there would be a type of a single element $q(x)$ over $\mathbb{Q}$ which does not split over $\emptyset$. But by quantifier elimination, every tuple of two distinct elements have the same type (i.e., $\mathrm{Aut}(\mathbb{Q}, C)$ acts 2-transitively on $\mathbb{Q}$). Now, q cannot be realized in $\mathbb{Q}$ and must contain $C(0, x, 1)$ or $C(1, x, 0)$, hence both, which is a contradiction. □

Remark 5.8. For any point $a \in \mathbb{Q}$, the expansion $(\mathbb{Q}, C, a)$ does have a CIR. Indeed, in this case C defines a dense linear order with no endpoints on $\mathbb{Q} \setminus \{a\}$ by $b < c \iff C(a, b, c)$. Since $(\mathbb{Q}, <)$ has a CIR $\mathop{\smile\!\!\!\!\!\mid}$ (Example 4.12), we can define $A \mathop{\smile\!\!\!\!\!\mid}^{*}_{C} B$ by $A \setminus \{a\} \mathop{\smile\!\!\!\!\!\mid}_{C \setminus \{a\}} B \setminus \{a\}$. Since for every finite tuples b, c, $b \equiv c$ in the expansion iff $b \setminus a \equiv c \setminus a$ in the order, it follows easily that $\mathop{\smile\!\!\!\!\!\mid}^{*}$ is a CIR.

An even closer look at the reducts of DLO, gives the following result.

Corollary 5.9. *Every closed supergroup of* $\mathrm{Aut}(\mathbb{Q}, <)$ *has topological rank* ≤ 3.

Proof. The diagram in [JZ08, page 867] of the lattice of closed groups between $\mathrm{Aut}(\mathbb{Q}, <)$ and $\mathrm{Aut}(\mathbb{Q}, =)$ shows that any such group contains at most two incomparable closed subgroups. Since no group can be a union of two of its proper subgroups, we do not need to use Neumanns's lemma in the proof of Theorem 5.4 above, allowing us to drop the assumption that (G, H) has no compact quotients. □

5.2.2. *A general reduction theorem.* In the next theorem we drop the assumption of having finitely many reducts of the expansion (i.e., of having finitely many groups between H and G), but we compensate for it by assuming that H acts oligomorphically on M and increasing the number of generators by 1.

Fact 5.10. [EH93, Lemma 1.4] *Suppose that M is a countable ω-saturated structure. Then for any $A, B \subseteq M$, there is some A' (in the monster model $\mathfrak{C}$, see just above Section 2.2) such that $A' \equiv A$ and $A' \cap B \subseteq \operatorname{acl}(\emptyset)$.*

Theorem 5.11. *Suppose as usual that M is countable and ω-categorical and let $G = \operatorname{Aut}(M)$. Suppose that $H \leq G$ is closed and acts oligomorphically on M and that (G, H) has no compact quotients. Then there are $g_1, g_2 \in G$ such that $H \cup \{g_1, g_2\}$ topologically generates G.*

Proof. Let M' be an ω-categorical expansion of M to some language L' containing L (the language of M) such that $H = \operatorname{Aut}(M')$. We use $'$ to indicate the expansion. In particular, $\mathfrak{C}'$ denotes the expansion of $\mathfrak{C}$ to L'.

By Fact 5.10, there is some M_0 such that $M_0 \equiv M$ and $M_0^{\mathrm{eq}} \cap M^{\mathrm{eq}} = \operatorname{acl}^{\mathrm{eq}}(\emptyset)$ (apply the fact in $\mathfrak{C}^{\mathrm{eq}}$). There is some automorphism σ of $\mathfrak{C}$ such that $\sigma(M_0) = M$. Let N_0' be a countable model containing $\sigma^n(M_0)$ for all $n \in \mathbb{Z}$. Let N_1' be a countable model containing $\sigma^n(N_0')$ for all $n \in \mathbb{Z}$. Continue like this and finally let $N_\omega' = \bigcup\{N_i' \mid i < \omega\}$. So $M' \prec N_\omega' \prec \mathfrak{C}'$ is countable and $\sigma \restriction N_\omega \in \operatorname{Aut}(N_\omega)$. By ω-categoricity (of M') we may assume that $N_\omega' = M'$: there is some $g_1 \in \operatorname{Aut}(M)$ and $M_0' \prec M'$ such that $g_1(M_0^{\mathrm{eq}}) \cap M_0^{\mathrm{eq}} = \operatorname{acl}^{\mathrm{eq}}(\emptyset)$.

Then $H_1 = \operatorname{cl}(\langle H, g_1\rangle)$ is a closed group acting oligomorphically on M. Also, note that (G, H_1) has no compact quotients. Let M'' be the reduct of M', which is also an expansion of M that corresponds to H_1: $\operatorname{Aut}(M'') = H_1$. As usual, we use $''$ to indicate that we work in this expansion.

Claim 5.12. If $X \subseteq M^n$ is definable over $\emptyset''$ (i.e., definable in L'' over $\emptyset$) and M-definable (in L), then it is $\emptyset$-definable (in L).

Proof. First note that it is enough to show that X is $\operatorname{acl}_L^{\mathrm{eq}}(\emptyset)$-definable (the code $\ulcorner X \urcorner$ of X belongs to $\operatorname{dcl}_{L''}^{\mathrm{eq}}(\emptyset)$ and to $\operatorname{acl}_L^{\mathrm{eq}}(\emptyset)$, and if it were not in $\operatorname{dcl}_L^{\mathrm{eq}}(\emptyset)$ then there would be an automorphism of M moving it, but then by the no-compact quotient assumption there would be an automorphism of M'' moving it as well — contradiction).

Now, since X is $\emptyset''$-definable and M-definable, it is definable over M_0 (because $M_0' \prec M'$), so its code $\ulcorner X \urcorner \in M_0^{\mathrm{eq}}$. In addition, $g_1(X) = X$, so X is definable over $g_1(M_0)$, hence $\ulcorner X \urcorner \in g_1(M_0^{\mathrm{eq}})$. Together it is in $\operatorname{acl}^{\mathrm{eq}}(\emptyset)$, which is what we wanted. □

Now we construct g_2 by back-and-forth to ensure that $\operatorname{cl}(\langle H_1, g_2\rangle) = G$.

Suppose that we have constructed $g_2 \restriction A$ for some finite set A. Let O be an orbit of the action of G on M^m, and we write it as $O = \bigcup \{O_i \mid i < n\}$ where the O_i's are the orbits of the action of H_1 (recall that H_1 acts oligomorphically on M, so there are only finitely many such orbits).

Claim 5.13. For any subset $s \subsetneq n$ there are $a, b \in O$ such that $a \in O_s = \bigcup \{O_i \mid i \in s\}, b \in O_{n \setminus s}$, and $g_2 \restriction A \cup \{\langle a, b\rangle\}$ or $g_2 \restriction A \cup \{\langle b, a\rangle\}$ is an elementary map.

Proof. Note that O_s is $\emptyset''$-definable. As it is not $\emptyset$-definable (because $s \subsetneq n$), it is also not M-definable by Claim 5.12. In particular, it is not A-definable. Hence there are $a_0 \in O_s, a_1 \in O_{n \setminus s}$ such that $a_0 \equiv_A a_1$. There is some b such that $a_0 A \equiv a_1 A \equiv b g_2 (A)$. If $b \in O_s$, then $g_2 \restriction A \cup \{\langle a_1, b\rangle\}$ is the required map. Otherwise, pick $g_2 \restriction A \cup \{\langle a_0, b\rangle\}$. □

In the back-and-forth construction of g_2, we deal with all these orbits (for every $m < \omega$, there are only finitely many) and all these subsets s and increase g_2 according to Claim 5.13. We claim that g_2 is such that $\mathrm{cl}(\langle H_1, g_2\rangle) = G$. Indeed, it is enough to show that every orbit O of G is also an orbit of $\langle H_1, g_2\rangle$. The orbit O can be written as $\bigcup \{O_i \mid i < n\}$ where the O_i's are the orbits of H_1, and also as $\bigcup \{O'_i \mid i \in I\}$ where the O'_i's are orbits of $\langle H_1, g_2\rangle$. Each such O'_i is itself a union of H_1-orbits, so has the form O_s for some $s \subseteq n$. But by construction, if $s \neq n$ there are tuples $a \in O_s, b \in O_{n \setminus s}$ such that either g_2 or g_2^{-1} maps a to b — contradiction. So $s = n$, and $O'_i = O$. □

6. A topological dynamics consequence of having a CIR

Definition 6.1. Suppose that M is a countable structure. Call an automorphism $\sigma \in G$ *shifty* if there is some invariant binary relation on finite sets in M, $\mathop{\smile\!\!\!\!\!\!|}$ (the base will always be $\emptyset$) such that:

- (Monotonicity) If $A \mathop{\smile\!\!\!\!\!\!|} B$ and $A' \subseteq A$, $B' \subseteq B$ then $A' \mathop{\smile\!\!\!\!\!\!|} B'$.
- (Right existence) For every finite tuple a there is some $a' \equiv a$ such that $a \mathop{\smile\!\!\!\!\!\!|} a'$ (by this we mean that sets enumerated by a, a' are independent).
- (Right shiftiness) If A is finite and b, b' are finite tuples such that $b' \equiv b$ and $A \mathop{\smile\!\!\!\!\!\!|} b'$, then there exists some $n < \omega$ such that $b' \equiv_A \sigma^n (b)$.

Lemma 6.2. *If σ is shifty then it also satisfies:*

- *(Left existence) For every finite tuple a there is some $a' \equiv a$ such that $a' \mathop{\smile\!\!\!\!\!\!|} a$.*
- *(Left shiftiness) If A is finite and b, b' are finite tuples such that $b' \equiv b$ and $b' \mathop{\smile\!\!\!\!\!\!|} A$, then there exists some $n < \omega$ such that $b' \equiv_A \sigma^{-n} (b)$.*

Proof. Suppose that σ is shifty, as witnessed by $\mathop{\smile\!\!\!\!\!\!|}$. Given a, there is some $a' \equiv a$ such that $a \mathop{\smile\!\!\!\!\!\!|} a'$. Applying an automorphism taking a' to a we get some $a'' \equiv a$ such that $a'' \mathop{\smile\!\!\!\!\!\!|} a$, which shows left existence.

As for left shiftiness, suppose that A is finite and enumerated by a, b, b' are finite tuples such that $b' \downarrow A$ and $b \equiv b'$. Then applying an automorphism, we get some a' such that $ab' \equiv a'b$, so $b \downarrow a'$. Hence for some $n < \omega$, $a' \equiv_b \sigma^n(a)$. From $a'b \equiv \sigma^n(a)\, b$ we get that $ab' \equiv a'b \equiv \sigma^{-n}(a')\,\sigma^{-n}(b) \equiv a\sigma^{-n}(b)$, i.e., $b' \equiv_A \sigma^{-n}(b)$. ☐

Proposition 6.3. *The automorphism σ is a shifty automorphism on M iff for any type $p \in S(\emptyset)$ (with finitely many variables), letting $Y_a = \bigcap\{\bigcup\{\mathrm{tp}(a, \sigma^n(a')) \mid n < \omega\} \mid a' \equiv a\}$ for any $a \models p$, the intersection $Y_p = \bigcap\{Y_a \mid a \models p\}$ is nonempty.*

Proof. Suppose that σ is shifty, and fix some type $p \in S(\emptyset)$. Let $a \models p$. By existence, there is some $a' \equiv a$ with $a \downarrow a'$. Let $q = \mathrm{tp}(a, a')$ and fix some $b \models p$. Let $\tau \in \mathrm{Aut}(M)$ map a to b and let $b' = \tau(a')$. We have that $b \downarrow b'$ and hence by right shiftiness, $q = \mathrm{tp}(b, b') \in Y_b$. Since b was arbitrary, $q \in Y_p$.

Suppose that the right hand side holds. Given a finite tuple a and $a' \equiv a$, write $a \downarrow^* a'$ iff $\mathrm{tp}(a, a') \in Y_p$ where $p = \mathrm{tp}(a/\emptyset)$. For general finite sets A, B, write $A \downarrow B$ iff there is some C containing A and C' containing B such that $C \downarrow^* C'$. Obviously, $\downarrow$ is invariant and monotone. Right existence follows from the assumption that $Y_p \neq \emptyset$ for all $p \in S(\emptyset)$. Right shiftiness: suppose that $a \downarrow^* a'$ and $a'' \equiv a'$. Then $\mathrm{tp}(a, a') \in Y_p$ and in particular it belongs to Y_a. By definition of Y_a, $\mathrm{tp}(a, a') \in \bigcup\{\mathrm{tp}(a, \sigma^n(a'')) \mid n < \omega\}$, so for some $n < \omega$, $aa' \equiv a_i\sigma^n(a'')$. ☐

Proposition 6.4. *If M is an ultrahomogeneous structure and $\downarrow$ is a CIR on finite subsets of M which respects substructures, then there exists a shifty automorphism σ on M, as witnessed by $\downarrow$.*

Proof. Monotonicity and right existence are parts of the properties of a CIR, so we only have to prove right shiftiness. Suppose that $A \downarrow b$ and $b' \equiv b$. By the proof of Theorem 3.12, the repulsive automorphism σ constructed there satisfies that for some $n < \omega$, $A \downarrow \sigma^n(b')$. By stationarity, $b \equiv_A \sigma(b')$. ☐

Recall the definitions of flow and subflow from Section 2.5.

Theorem 6.5. *Let M be a countable homogeneous structure and $G = \mathrm{Aut}(M)$.*

Suppose that $\sigma \in G$ is a shifty automorphism and that (X, d) is a compact metric G-flow. Then for every $x_* \in X$ there is some conjugate $\sigma_* \in G$ of σ such that:

(*) *Both $\mathrm{cl}\{\sigma_*^n(x_*) \mid n < \omega\}$ and $\mathrm{cl}\{\sigma_*^{-n}(x_*) \mid n < \omega\}$ contain a subflow of X.*

Remark 6.6. Note that Theorem 6.5 implies that both $\bigcap\{\mathrm{cl}\{\sigma_*^n(x_0) \mid k \leq n < \omega\} \mid k < \omega\}$ and $\bigcap\{\mathrm{cl}\{\sigma_*^{-n}(x_0) \mid k \leq n < \omega\} \mid k < \omega\}$ contain a subflow of X: if e.g., Y_0 is a flow contained in the left space, then $GY_0 = Y_0$, so $\sigma_*^{-k}(Y_0) \subseteq Y_0 \subseteq \mathrm{cl}\{\sigma_*^n(x_*) \mid n < \omega\}$, hence $Y_0 \subseteq \sigma_*^k(\mathrm{cl}\{\sigma_*^n(x_*) \mid n < \omega\}) = \mathrm{cl}\{\sigma_*^n(x_*) \mid k \leq n < \omega\}$.

Before the proof we note the following useful lemma.

Lemma 6.7. *Suppose that G is a topological group acting continuously on a compact metric space (X,d). Then for every $0<\varepsilon$ there is some open neighborhood U of $\mathrm{id}\in G$ such that for every $g,h\in G$ if $gh^{-1}\in U$ then for all $x\in X$ we have that $d(gx,hx)<\varepsilon$.*

Proof. It is enough to show that there is some open neighborhood U of id such that if $g\in U$ then for all $x\in X$, $d(gx,x)<\varepsilon$ (since then if $gh^{-1}\in U$ then $d\left(gh^{-1}(hx),hx\right)<\varepsilon$). For every $x\in X$, there is some neighborhood V_x of x in X and some neighborhood U_x of id in G such that for all $g\in U_x$, $x'\in V_x$, $d(gx',x')<\varepsilon$. By compactness, a finite union of V_x's covers X. Let U be the intersection of the corresponding U_x's. □

Proof of Theorem 6.5. Suppose that $\downarrow$ witnesses that σ is shifty. Let G_0 be a countable dense subset of G, enumerated as $\langle g_i \mid i<\omega\rangle$, such that $g_0=\mathrm{id}$.

We construct an automorphism $\tau:M\to M$ by back and forth such that eventually $\sigma_*=\tau^{-1}\sigma\tau$ and such that at each finite stage, τ will be an elementary map. For the construction it is actually better to think of the domain and range of τ as two different structures, so we have $M=M_*$ and suppose that $\sigma:M\to M$, $\sigma_*:M_*\to M_*$ and $\tau:M_*\to M$. The subscript $*$ will denote tuples from M_* throughout.

Suppose that we have constructed a partial elementary map $f:A_*\to A$ (that will be part of τ eventually) with $A_*\subseteq M_*, A\subseteq M$ finite, enumerated by a_*,a. Here is the main tool in the construction.

Claim 6.8. Suppose that $b'_*\downarrow a_*$ and $b'_*\equiv b\subseteq a$. Let $b_*=f^{-1}(b)$. Then there is $k<\omega$ and an extension f' of f such that any automorphism τ' extending f' will satisfy that for $\sigma'_*=\tau'^{-1}\sigma\tau'$, $\sigma'^k_*(b'_*)=b_*$.

Similarly, if $a_*\downarrow b'_*$ then there is some $k<\omega$ and an extension f' of f such that any automorphism τ' extending f' will satisfy that for $\sigma'_*=\tau'^{-1}\sigma\tau'$, $\sigma'^k_*(b_*)=b'_*$.

Proof. First, find some tuple b' in M such that $b'a\equiv b'_*a_*$. In particular, $b'\downarrow a$. By left shiftiness, there is some $k<\omega$ such that $\sigma^{-k}(b)\,a\equiv b'a\equiv b'_*a_*$. Extend f to f' which sends b'_* to $\sigma^{-k}(b)$. Then, for any τ' extending f', $\tau'^{-1}\sigma^k\tau'(b'_*)=\tau'^{-1}(b)=b_*$.

The second statement is proved similarly, using right shiftiness. □

We will make sure that for each $n<\omega$, the following condition holds.

- $\star$ There are $k_{n,0},\dots,k_{n,n-1}<\omega$ such that for all $i<n$, $d\left(\sigma_*^{k_{n,i}}(x_*),g_i\left(\sigma_*^{k_{n,0}}(x_*)\right)\right)<1/n$ and $k'_{n,0},\dots,k'_{n,n-1}<\omega$ such that for all $i<n$, $d\left(\sigma_*^{-k'_{n,i}}(x_*),g_i\left(\sigma_*^{-k'_{n,0}}(x_*)\right)\right)<1/n$.

Why is $\star$ enough? Let $y_n=\sigma_*^{k_{n,0}}(x_*)$, and let y be a limit of some subsequence $\langle y_{n_j}\mid j<\omega\rangle$ (which exists by compactness), then $Gy\subseteq \mathrm{cl}\{\sigma_*^n(x_*)\mid n<\omega\}$ (so $\mathrm{cl}(Gy)$ is a subflow): given $g\in G$ and $0<\varepsilon$, first find an open neighborhood $U\subseteq G$ of g such that if $h\in U$ then $d(gx,hx)<\varepsilon/4$ for all $x\in X$ (this U is given to us by Lemma 6.7: it is $\left(g^{-1}V\right)^{-1}$ where V is an open neighborhood of id such that if $gh^{-1} \in V$, $d(gx, hx) < \varepsilon/4$). Take n so large that $g_i \in U$ for some $i < n$ and $1/n < \varepsilon/4$, and find n_j even larger so that $d\left(g_i y, g_i y_{n_j}\right) < \varepsilon/4$. Then $d(gy, g_i y) < \varepsilon/4$, $d\left(g_i y, g_i y_{n_j}\right) < \varepsilon/4$ and $d\left(g_i y_{n_j}, \sigma_*^{k_{n_j,i}}(x_*)\right) < \varepsilon/4$. Together, $d\left(gy, \sigma_*^{k_{n_j,i}}(x_0)\right) < 3\varepsilon/4 < \varepsilon$, which means that gy is in the closure. Similarly, if y' is a limit of a subsequence of $\sigma_*^{-k'_{n,0}}(x_*)$, then $Gy' \subseteq \mathrm{cl}\{\sigma_*^{-n}(x_*) \,|\, n < \omega\}$.

So we consider $f : A_* \to A$ a partial elementary map. Our task now is to deal with $n < \omega$. Let $\varepsilon = 1/n$.

Let $A_* \subseteq C_* \subseteq M_*$ be finite such that if $g^{-1} \restriction C_* = h^{-1} \restriction C_*$ then $d(gx, hx) < \varepsilon/4$ for all $x \in X$ and any $g, h \in G$ (this is by Lemma 6.7). Let $z_0, \ldots, z_{l-1}$ be such that $\bigcup \{B(z_j, \varepsilon/4) \,|\, j < l\}$ cover X, and write $B_j = B(z_j, \varepsilon/4)$.

Let c_* be a finite tuple enumerating C_*. For every $c'_* \equiv c_*$, we say that c'_* *has color* $j < l$ if j is least such that there is $g \in G$ such that $g(c'_*) = c_*$ and $gx_* \in B_j$. Note that by the choice of c_*, if $g'(c'_*) = c_*$ then $g'g^{-1} \restriction C_* = \mathrm{id}$, so $g'x_0 \in B(z_j, \varepsilon/2)$.

Let $D_* = \bigcup \left\{g_i^{-1}(C_*) \,\middle|\, i < n\right\}$. Note that $C_* \subseteq D_*$ because $g_0 = \mathrm{id}$. Let d_* enumerate D_*. For any $d'_* \equiv d_*$ and $s \subseteq l$, we say that d'_* *has color* s if $\{j < l \,|\, c'_* \equiv c_*, c'_* \subseteq d'_*, c'_* \text{ has color } j\} = s$.

By left existence, there is some $s_0 \subseteq l$ such that for *every* finite set $S \subseteq M_*$, there is some $d'_* \equiv d_*$ with $d'_* \mathop{\smile\!\!\!\!\!\mid}\, S$ and d'_* has color s_0.

Let $d'_* \equiv d_*$ be of color s_0 such that $d'_* \mathop{\smile\!\!\!\!\!\mid}\, d_*$. For $i < n$, let $c'_{*,i} \subseteq d'_*$ be the tuple corresponding to $g_i^{-1}(c_*)$, so in particular $c'_{*,i} \equiv c_*$. Let $j_i < l$ be the color of $c'_{*,i}$. By the choice of s_0, for every finite set $S \subseteq M_*$, there is some $c'_* \equiv c_*$ such that $c'_* \mathop{\smile\!\!\!\!\!\mid}\, S$ and c'_* has color j_i.

Since M is homogeneous we can extend f in such a way so that its domain equals D_*. By Claim 6.8 (the first part), there is some $k_{n,0} < \omega$ and an extension f_0 of f that ensures that $\sigma_*^{k_{n,0}}(d'_*) = d_*$.

Starting with f_0, we construct an increasing sequence $\langle f_i \,|\, i < n\rangle$ as follows. Suppose we have f_i whose domain is $D_{*,i}$. Find some $c''_{*,i+1} \equiv c_*$ of color j_{i+1} such that $c''_{*,i+1} \mathop{\smile\!\!\!\!\!\mid}\, D_{*,i}$. By Claim 6.8, we can find $k_{n,i+1} < \omega$ and extend f_i to f_{i+1} which ensures that $\sigma_*^{k_{n,i+1}}\left(c''_{*,i+1}\right) = c_*$.

Now we have the first part of $\star$: we need to check that $d\left(\sigma_*^{k_{n,i}}(x_*), g_i\left(\sigma_*^{k_{n,0}}(x_*)\right)\right) < \varepsilon$ for all $i < n$. For $i = 0$ this is clear since $g_0 = \mathrm{id}$, so we may assume that $i > 0$. As $\sigma_*^{k_{n,i}}\left(c''_{*,i}\right) = c_*$, it follows that $d\left(\sigma_*^{k_{n,i}}(x_0), z_{j_i}\right) < \varepsilon/2$. Similarly, as $\sigma_*^{k_{n,0}}\left(c'_{*,i}\right) = g_i^{-1}(c_*)$, we have that $d\left(g_i\left(\sigma_*^{k_{n,0}}(x_0)\right), z_{j_i}\right) < \varepsilon/2$. Together, we are done.

Now we have to take care of the other half of $\star$. This is done similarly, using right existence and the second part of Claim 6.8. □

The following proposition explains why we needed to take a conjugate of σ. The countable ordered random graph has a CIR by Example 4.16, thus Theorem 6.5 applies to it. In Section

2.5, we mentioned that it is a Ramsey structure. Note that the underlying order is dense (by Proposition 2.4).

Proposition 6.9. *Let $M = (V, <, R)$ be the countable ordered random graph. Then there is no automorphism $\sigma \in G = \mathrm{Aut}(M)$ which satisfies (*) for every continuous action on a compact metric space X on which G acts and every $x_* \in X$.*

Proof. First we find $a \neq b$ in M such that $\sigma^n(a) \neq \sigma^m(b)$ for all $m, n \in \mathbb{Z}$. To do that, take any $a \in M$. Then $\{\sigma^n(a) \mid n \in \mathbb{Z}\}$ is discrete (in the order sense: it is either a $\mathbb{Z}$-chain or just a). Since $(V, <)$ is dense, there is some $b \neq \sigma^n(a)$ for all $n \in \mathbb{Z}$. It follows that b is as required. Let $X = S_x(M)$ be the space of complete types over M (in one variable x) (it is a compact metric space). Let $p \in X$ be any completion of the partial type $\{R(x, \sigma^n(a)) \mid n \in \mathbb{Z}\} \cup \{\neg R(x, \sigma^m(b)) \mid m \in \mathbb{Z}\}$. Then if (*) holds for p, then by Fact 2.20, there is some point $p_0 \in \mathrm{cl}\{\sigma^n(p) \mid n < \omega\}$ which is a fixed point of G. In other words, p_0 is an invariant type over M. However $R(x, a) \wedge \neg R(x, b) \in p_0$ (this is true for any type in the closure), so p_0 cannot be invariant (because G is transitive). □

The example of the ordered random graph also explains why we needed to restrict to compact metric spaces, and could not prove this for all compact spaces. If Theorem 6.5 had worked for all compact spaces, it would also work for the universal G-ambit (see Section 2.5), (X, x_0). Thus, there would be a conjugate σ_* of σ such that $\mathrm{cl}\{\sigma_*^n(x_0) \mid n < \omega\}$ contains a subflow. But then if (Y, y_0) is any other G-ambit, by universality, there is a continuous surjection $\pi : X \to Y$ mapping x_0 to y_0 and commuting with the action of G. Thus, π maps $\mathrm{cl}\{\sigma_*^n(x_0) \mid n < \omega\}$ to $\mathrm{cl}\{\sigma_*^n(y) \mid n < \omega\}$, and the latter contains a G-subflow. Thus we get that σ_* satisfies (*) for every G-ambit, which contradicts Proposition 6.9.

Corollary 6.10. *Let $T = T_{dt}$ be the theory of dense trees in the language $\{<, \wedge\}$, and let $M \models T$ be countable. Then $\mathrm{Aut}(M)$ has no shifty automorphism. In particular, M has no CIR.*

Furthermore, the same is true for $T_{dt,<_{lex}}$, the theory of the lexicographically ordered dense tree N, see Example 2.16.

Proof. Suppose that σ was shifty. Let $\bar{m} = \langle m \mid m \in M \rangle$ be an enumeration of M (really the identity function), and let $\bar{x} = \langle x_m \mid m \in M \rangle$. Let $X = S_{\bar{m}}(M)$ be the space of $\bar{x}$-complete types p over M such that $p \upharpoonright \emptyset = \mathrm{tp}(\bar{m}/M)$. Then X is a compact metric space. Let $x_* = \mathrm{tp}(\bar{m}/M)$. By Theorem 6.5, there is some conjugate τ of σ such that $\mathrm{cl}\{\tau^n(x_*) \mid n < \omega\}$ contains a subflow $Y^+ \subseteq X$ and similarly, $\mathrm{cl}\{\tau^{-n}(x_*) \mid n < \omega\}$ contains a subflow Y^-. By Proposition 4.19, τ fixes a branch or a point.

Suppose that $\tau(m) = m$ for some $m \in M$. Then for every $p \in Y^+$, $p \models x_m = m$. However $G = \mathrm{Aut}(M)$ acts transitively on M, so we have a contradiction.

Now suppose that τ fixes a branch $B \subseteq M$, but does not fix any point. Suppose that $\tau(m) > m$ for some $m \in B$. Then $\tau^n(m) > m$ for all $n < \omega$, so for any $p \in Y^+$, $p \models x_m > m$. There is some $m' \in M$ such that $m' > m$ and $m' \notin B$. Since $m < \tau^n(m) \in B$ for all $n < \omega$, it follows that $p \models x_m \wedge m' = m$ for all $p \in Y^+$. Let $\tau' \in G$ fix m and map m' to B. Then $\tau'(p) \models (x_m \wedge \tau'(m')) = m < x_m$. But $\tau'(p) \in Y^+$, so $\tau'(p) \models x_m \leq \tau'(m') \vee \tau'(m') \leq x_m$, which is a contradiction. If, on the other hand $\tau(m) < m$, then $\tau^{-1}(m) > m$, so we can apply the same argument to Y^-.

For the furthermore part, note that by Proposition 2.4, the reduct of $T_{dt,<_{lex}}$ to the tree language is T_{dt}. In addition, letting $H = \mathrm{Aut}(N)$, H acts transitively on N (by quantifier elimination, as N is ultrahomogeneous). In addition, if $B \subseteq N$ is a branch, $m \in B$, there is always some $m' > m$, $m' \notin B$ and for any $n' > m$ in B, $m'm \equiv n'm$. Hence, we can apply Proposition 4.19 and the same proof will work. □

7. Further questions

The results presented in the previous sections lead to a number of questions, both related to CIR and more generally on ω-categorical structures. We state here a few general conjectures and questions. If they turn out to be false at this level of generality, they could be weakened by restricting to finitely homogeneous structures or other subclasses.

The following conjecture, along with Theorem 5.11 (and Example 2.12), would imply that indeed compact quotients are the only obstruction to having finite topological rank.

Conjecture 7.1. *Any ω-categorical structure has an ω-categorical expansion which admits a CIR.*

Suppose that M is a structure and $\mathfrak{C}$ a monster model for $\mathrm{Th}(M)$. The group of *Lascar strong automorphisms of* M, denoted by $\mathrm{Aut}\,f(M)$ is the group of automorphisms of M generated by the set $\{\sigma \restriction M \mid \exists N \prec \mathfrak{C}, |N| = |T|, \sigma \restriction N = \mathrm{id}\}$. If σ is Lascar strong, then $\sigma \restriction \mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{id}$ so $\mathrm{Aut}\,f(M)$ is contained in G^0. However, there are examples (even ω-categorical examples) where G^0 is strictly bigger than $\mathrm{Aut}\,f(M)$, see [Iva10, Pel08]. The *Lascar group* is the quotient $\mathrm{Aut}(M) / \mathrm{Aut}\,f(M)$. For more on the Lascar group, see [Zie02]. In the ω-categorical case, the quotient $\mathrm{Aut}(\mathrm{acl}^{\mathrm{eq}}(\emptyset))$ is also called the *compact Lascar group.*

If M is an ultrahomogeneous linearly ordered Ramsey structure, then by (the proof of) Proposition 3.17, there is some model N such that $N \mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$\mid$}}^{ns} M$. In particular, $\sigma(M) \equiv_N M$, for every $\sigma \in \mathrm{Aut}(M)$ which implies that σ is Lascar strong . Thus in Ramsey structures, and in fact for any model M for which there is some such N, the Lascar group is trivial, and there are no compact quotients. For instance, by Lemma 3.15 this happens also when M is ω-categorical with a CIR.

During a talk given on this paper by the second author, Anand Pillay asked whether the Lascar group could be an obstruction to finite topological rank. A positive answer to the following would imply that it is not (since, given this conjecture, if $\mathrm{Aut}(M)$ has no compact quotients, then let M' be an expansion with trivial Lascar group. If we knew that M' has a CIR, then we could apply Theorem 5.11).

Conjecture 7.2. *Any ω-categorical structure has an ω-categorical expansion with trivial Lascar group.*

By the above, this second conjecture is implied by Conjecture 7.1.

Note also that by Proposition 2.13, the conjecture is true when we replace the Lascar group by the compact Lascar group.

It would be interesting to investigate other consequences of having a CIR. For instance a CIR might have something to say about normal subgroups. The analysis in [DHM89] of automorphism groups of trees seems to suggest that there is a link: normal subgroups appear as groups fixing a set of points roughly corresponding to the set of x such that $x \mathop{\smile\!\!\!\!\!\!\,|}\, A$ for some CIR $\mathop{\smile\!\!\!\!\!\!\,|}$ and finite set A. A similar phenomenon happens in DLO, where there are only three normal subgroups (the group of automorphism fixing a cone to the left, to the right, and the intersection of these two), see [Gla81, Theorem 2.3.2].

In another direction, recall that an automorphism group G (or more generally a Polish group) has the *small index property* (sip) if every subgroup of index less than $2^{\aleph_0}$ is open. Many groups are known to have this property, but there are at least two different types of techniques used to show it—the Hrushovski property (or extension property) and direct combinatorial methods—which have yet to be unified. We refer to [Mac11] for a survey on this. As in the case of finite topological rank, large compact quotients seem to be only known obstruction to having sip, although the situation is more complicated: Lascar [Las91, end of Section 2.2] gives an example of an automorphism group without the sip and with no compact quotients. In fact the compact quotients are hidden in the stabilizer of a finite set. It seems that one can avoid this counterexample by restricting to dense subgroups. This leads us to the following questions.

Question 7.3. *Let M be ω-categorical such that $G = \mathrm{Aut}(M)$ has no compact quotient. Is it true that any dense subgroup of G of index less than $2^{\aleph_0}$ is open (and hence is equal to G)?*

Note that the assumption of having no compact quotient is necessary. Indeed, in the example suggested by Cherlin and Hrushovski (the one described in Remark 3.2), we have that $G = \mathrm{Aut}(M)$ has a dense subgroup of index 2, see [Las91, Section 2.1].

Question 7.4. *Let M be ω-categorical and N an ω-categorical expansion of M. Set $G = \mathrm{Aut}(M)$ and $H = \mathrm{Aut}(N) \leq G$. Assume that (G, H) has no compact quotients and that H has the sip. Is it true that any dense subgroup of G of index less than $2^{\aleph_0}$ is open?*

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